THE CANONICAL GROTHENDIECK TOPOLOGY AND A HOMOTOPICAL ANALOG

by

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DISSERTATION ABSTRACT

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We explore the canonical Grothendieck topology and a new homotopical analog. First we discuss a specific description of the covers in the canonical topology, which we then use to get a corollary of Giraud's Theorem. Second we delve into the canonical topology on some specific categories, e.g. on the category of topological spaces and the category of abelian groups; this part includes concrete examples and non-examples. Lastly, we discuss a homotopical analog of the canonical Grothendieck topology and explore some examples of this analog.

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CHAPTER I

INTRODUCTION

In SGA 4.2.2 Verdier introduced the canonical Grothendieck topology. He defined the *canonical topology* on a category $\mathcal{C}$ to be the largest Grothendieck topology where all representable presheaves are sheaves. With such an implicit definition we naturally start to wonder how one can tell what collection of maps are or are not in the canonical topology. In order to obtain a more explicit description of the canonical topology we define a notion of 'universal colim sieve,' prove that the collection of all univeral colim sieves forms a Grothendieck topology, and then prove that this topology is precisely the canonical topology. Furthermore, we explore the notation of universal colim sieve in greater depth for the categories of sets, topological spaces and R-modules; this exploration sometimes yields another explicit description of the canonical topology.

But what if the category has extra structure (e.g. what if $\mathcal{C}$ is a model category)? In this situation, one might wonder how the canonical topology interacts with this extra structure or if there is an analog of the canonical topology that utilizes the additional structure. We explore the later by defining an analog of the canonical topology in the homotopical setting by using weak equivalences and homotopy colimits, proving that our analog is a Grothendieck topology for any simplicial model category, and showcasing some examples.

We start by spending some time exploring background results and definitions, some of which will make computations and examples easier and some of which are general results (e.g. about effective epimorphisms) that will be used in later proofs.

Then we prove that the canonical topology can be described using universal colim sieves and examine the implications of this description. One implication is that this description allows us to prove (as a corollary to Giraud's Theorem) that every category $\mathcal{C}$, which satisfies some hypotheses, is equivalent to the category of sheaves on $\mathcal{C}$ with the canonical topology. As another implication, we use this universal colim sieve description to investigate the canonical topology on the categories of sets, topological spaces and R-modules. In our exploration of the canonical topology on the category of topological spaces we are able to refine our description and obtain a basis for the canonical topology; this result reduces the question "is this in the canonical topology?" to the question "is a specific map a universal quotient map?" Since universal quotient maps have been studied in-depth (for example by Day and Kelly in [2]), this reduction becomes our most computationally agreeable description of the canonical topology and hence we use it to find some specific examples and non-examples. In our investigation of the canonical topology on the category of abelian groups we work towards refining our description by making some reductions and obtaining some exclusionary results. While these reductions and results lead us to some specific examples and non-examples, a basis for the canonical topology remains elusive.

Furthermore, our definition of universal colim sieve makes the following statement automatic: a universal colim sieve is generated by one morphism if and only if that morphism is a universal effective epimorphism. Thus we obtain a connection between universal effective epimorphisms and the canonical topology, but this is not the first time the phrase 'universal effective epimorphism' has been linked to a Grothendieck topology. In [14] Quillen defines a Grothendieck topology by declaring S to be a cover if and only if S contains a universal effective epimorphism.

This connection may make one wonder if Quillen's topology is in fact the canonical topology. We will briefly discuss Quillen's topology and its relationship with the canonical topology.

Lastly, we take the homotopical analog of our canonical topology's "explicit" description. By generalizing some of our proofs relating to the canonical topology, we obtain a proof that this analog also forms a Grothendieck topology. We finish by exploring some examples of this analog in the category of topological spaces.

Sieves will be of particular importance in this paper and so we start with a reminder of their definition and a reminder of the definition of a Grothedieck topology (in terms of sieves); both definitions follow the notation and terminology used by Mac Lane and Moerdijk in [10].

For any object X of a category $\mathcal{C}$, we call S a *sieve on* X if S is a collection of morphisms, all of whose codomains are X, that is closed under precomposition, i.e. if $f \in S$ and $f \circ g$ makes sense, then $f \circ g \in S$. In particular, we can view a sieve S on X as a full subcategory of the overcategory $(\mathcal{C} \downarrow X)$.

A *Grothendieck topology* is a function that assigns to each object X a collection $J(X)$ of sieves such that

1. (Maximality) $\{f \mid \text{codomain } f = X\} = (\mathcal{C} \downarrow X) \in J(X)$
2. (Stability) If $S \in J(X)$ and $f\colon Y \to X$ is a morphism in $\mathcal{C}$, then
   $f^*S := \{g \mid \text{codomain } g = Y,\ f \circ g \in S\} \in J(Y)$
3. (Transitivity) If $S \in J(X)$ and R is any sieve on X such that
   $f^*R \in J(\text{domain } f)$ for all $f \in S$, then $R \in J(X)$.

Organization. In Chapter II, we go over some background information and define (universal) colim sieves; this includes some basic results in 2.1 that will make computations and proofs easier. Sections 2.2 and 2.3 cover some terminology and results that are needed for the proofs of Theorems 3.1.1 and 8.0.2. In Section 3.1 we prove that the collection of all universal colim sieves forms a Grothendieck topology, which in Section 3.2 we prove is the canonical topology. In Section 3.2.1 we give a basis for the canonical topology on a very specific type of category. Then in Chapter IV we prove a corollary of Giraud's Theorem. In Chapter V we look at the canonical topology on the category of sets and the category of topological spaces; this includes discussing specific examples in both the category of all topological spaces and a "convenient subcategory." In Chapter VI we look at the canonical topology on the category of R-modules and the category of abelian groups; this includes reductions and discussing specific examples and non-examples. In Chapter VII we discuss the Quillen topology and how it relates to the canonical topology. In Chapter VIII we discuss the homotopical versions of (universal) colim sieves and the canonical Grothendieck topology. Lastly, in Chapter IX, we discuss some examples of the homotopical version of universal colim sieves.

General Notation.

Notation 1.0.1. For any subcategory S of $(\mathcal{C} \downarrow X)$, we will use U to represent the forgetful functor $S \to \mathcal{C}$. For example, for a sieve S on X, $U(f) = \text{domain } f$.

Notation 1.0.2. For any category $\mathcal{D}$ and any two objects P, M of $\mathcal{D}$, we will write $\mathcal{D}(P, M)$ for $\mathrm{Hom}_{\mathcal{D}}(P, M)$.

Notation 1.0.3. We say that a sieve S on X is *generated* by the morphisms $\{f_\alpha \colon A_\alpha \to X\}_{\alpha \in \mathcal{A}}$ and write $S = \langle \{f_\alpha \colon A_\alpha \to X\}_{\alpha \in \mathcal{A}} \rangle$ if each $f \in S$ factors through one of the f_α, i.e. if $f \in S$ then there exists an $\alpha \in \mathscr{A}$ and morphism g such that $f = f_\alpha \circ g$.

CHAPTER II

BACKGROUND INFORMATION

This section contains the preliminaries for the rest of the document, starting with the following important definitions:

Definition 2.0.1. For a category $\mathcal{C}$, an object X of $\mathcal{C}$ and sieve S on X, we call S a *colim sieve* if $\varinjlim_S U$ exists and the canonical map $\varinjlim_S U \to X$ is an isomorphism. (Alternatively, S is a colim sieve if X is the universal cocone under the diagram $U \colon S \to \mathcal{C}$.) Moreover, we call S a *universal colim sieve* if for all arrows $\alpha \colon Y \to X$ in $\mathcal{C}$, $\alpha^* S$ is a colim sieve on Y.

Remark 2.0.2. In [6] Johnstone also defined sieves of this form but the term 'effectively-epimorphic' was used instead of the term 'colim sieve.'

2.1 Basic Results

This section mentions some basic results, all of which we believe are well-known folklore but we include them here for completeness.

Lemma 2.1.1. Suppose $\mathcal{C}$ is a category with all pullbacks.
Let $S = \langle \{g_\alpha \colon A_\alpha \to X\}_{\alpha \in \mathfrak{A}} \rangle$ be a sieve on object X of $\mathcal{C}$ and $f \colon Y \to X$ be a morphism in $\mathcal{C}$. Then $f^* S = \langle \{A_\alpha \times_X Y \xrightarrow{\pi_2} Y\}_{\alpha \in \mathfrak{A}} \rangle$.

Proof. It is an easy exercise. $\square$

Proposition 2.1.2. Let $\mathcal{C}$ be a cocomplete category. For $S = \langle \{f_\alpha \colon A_\alpha \to X\}_{\alpha \in \mathfrak{A}} \rangle$ a sieve in $\mathcal{C}$ on X such that $A_i \times_X A_j$ exists for all $i, j \in \mathfrak{A}$,

$$\varinjlim_S U \cong \mathrm{Coeq}\begin{pmatrix} \displaystyle\coprod_{(i,j)\in\mathfrak{A}\times\mathfrak{A}} A_i \times_X A_j \\ \downarrow\downarrow \\ \displaystyle\coprod_{k\in\mathfrak{A}} A_k \end{pmatrix}$$

where the left and right vertical maps are induced from the projection morphisms $\pi_1 \colon A_i \times_X A_j \to A_i$ and $\pi_2 \colon A_i \times_X A_j \to A_j$.

Proof. Let I be the category with objects α and (α, β) for all $\alpha, \beta \in \mathfrak{A}$ and unique non-identity morphisms $(\alpha, \beta) \to \alpha$ and $(\alpha, \beta) \to \beta$. Define a functor $L \colon I \to S$ by $L(\alpha) = f_\alpha$ and $L(\alpha, \beta) = f_{\alpha,\beta}$ where $f_{\alpha,\beta} \colon A_\alpha \times_X A_\beta \to X$ is the composition $f_\alpha \circ \pi_1 = f_\beta \circ \pi_2$. It is an easy exercise to see that L is final in the sense that for all $f \in S$ the undercategory $(f \downarrow L)$ is connected. Thus by [9, Theorem 1, Section 3, Chapter IX]

$$\varinjlim_S U \cong \varinjlim_I UL.$$

But by the universal property of colimits, $\varinjlim_I UL$ is precisely the coequalizer mentioned above.

$\square$

Lemma 2.1.3. Let $\mathcal{C}$ be a category. Then S is a colim sieve on X if and only if f^*S is a colim sieve for any isomorphism $f \colon Y \to X$.

Proof. It is an easy exercise. $\square$

Recall that a morphism $f \colon Y \to X$ is called an *effective epimorphism* provided $Y \times_X Y$ exists, f is an epimorphism and $c \colon \mathrm{Coeq}\,(Y \times_X Y \rightrightarrows Y) \to X$ is an isomorphism. Note that this third condition actually implies the second because $f = c \circ g$ where $g\colon Y \to \operatorname{Coeq}(Y \times_X Y \rightrightarrows Y)$ is the canonical map. Indeed, g is an epimorphism by an easy exercise and c is an epimorphism since it is an isomorphism.

Additionally, $f\colon Y \to X$ is called a *universal effective epimorphism* if f is an effective epimorphism with the additional property that for every pullback diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\pi_g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

π_g is also an effective epimorphism.

Remark 2.1.4. A morphism $f\colon A \to B$ is called a *regular epimorphism* if it is a coequalizer of some pair of arrows. When the pullback $A \times_B A$ of f exists in the category $\mathcal{C}$, then it is easy to see that f is a regular epimorphism if and only if f is an effective epimorphism.

Corollary 2.1.5. Let $\mathcal{C}$ be a cocomplete category with pullbacks. If

$$S = \langle \{ f\colon Y \to X \} \rangle$$

is a sieve on X, then S is a colim sieve if and only if f is an effective epimorphism. Moreover, S is a universal colim sieve if and only if f is a universal effective epimorphism.

Proof. The condition for f to be an effective epimorphism is, by Proposition 2.1.2, precisely what it means for S to be a colim sieve. $\square$

2.1.1 Effective Epimorphisms

Now we take a detour away from (universal) colim sieves to discuss some results about effective epimorphisms, which will be used in the proof of Theorem 3.2.4. We start with a terminology reminder [see 7]: we call $f\colon A \to B$ a *strict epimorphism* if any morphism $g\colon A \to C$ with the property that $gx = gy$ whenever $fx = fy$ for all D and $x, y\colon D \to A$, factors uniquely through f, i.e. $g = hf$ for some unique $h\colon B \to C$.

Proposition 2.1.6. If the category $\mathcal{C}$ has all pullbacks, then a morphism f is an effective epimorphism if and only if f is a strict epimorphism.

Proof. Let $f\colon A \to B$ be our morphism. First suppose that f is an effective epimorphism. Let $g\colon A \to C$ be a morphism with the property that $gx = gy$ whenever $fx = fy$. Since f is an effective epimorphism, then the commutative diagram

$$\begin{array}{ccc} A \times_B A & \xrightarrow{\pi_1} & A \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ A & \xrightarrow[f]{} & B \end{array}$$

is both a pushout and pullback diagram. Since the diagram is commutative, i.e. $f\pi_1 = f\pi_2$, then $g\pi_1 = g\pi_2$. Now the universal property of pushouts implies that there exists a unique $h\colon B \to C$ such that $g = hf$. Hence f is a strict epimorphism.

To prove the converse, suppose that f is a strict epimorphism. Consider the diagram

$$\mathcal{F} := \left\{ \; A \times_B A \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \; \right\}.$$

We will show that B is Coeq($\mathcal{F}$) by showing that B satisfies the univeral property of colimits with respect to $\mathcal{F}$. Specifically, suppose we have a morphism $\mathcal{F} \to C$, i.e. there is a morphism $g\colon A \to C$ such that $g\pi_1 = g\pi_2$.

Suppose we know $gx = gy$ whenever $x, y\colon D \to A$ and $fx = fy$. Then, since f is strict, this implies that there exists a unique $h\colon B \to C$ such that $g = hf$. Hence, B satisfies the universal property of colimits and so $B \cong \operatorname{Coeq} \mathcal{F}$.

Thus to show that f is an effective epimorphism, it suffices to show:

$$\text{if } x, y\colon D \to A \text{ and } fx = fy, \text{ then } gx = gy.$$

For a fixed pair $x, y\colon D \to A$ such that $fx = fy$, we have the commutative diagram

$$\begin{tikzcd} D \arrow[r, "x"] \arrow[d, "y"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Thus, by the universal property of pullbacks, both x and y factor through the pullback $A \times_B A$, i.e. $x = \pi_1\alpha$ and $y = \pi_2\alpha$ for some unique morphism $\alpha\colon D \to A \times_B A$. Therefore, our assumption $g\pi_1 = g\pi_2$ implies

$$gx = g\pi_1\alpha = g\pi_2\alpha = gy.$$

Hence g has the property that $gx = gy$ whenever $fx = fy$. $\square$

Corollary 2.1.7. If the category $\mathcal{C}$ has all pullbacks, then universal effective epimorphisms are closed under composition.

Proof. In [8, Proposition 5.11] Kelly proves that totally regular epimorphisms are closed under composition; our Corollary follows immediately from Kelly's result and our Proposition 2.1.6. We will end with a few remarks: what Kelly called regular epimorphisms are what we are calling strict epimorphisms, and Kelly's *totally* condition is precisely our *universal* condition. $\square$

Before our next result, we review some definitions. Let $\mathcal{E}$ be a category with small hom-sets, all finite limits and all small colimits. Let E_α be a family of objects in $\mathcal{E}$ and $E = \amalg_\alpha E_\alpha$.

The coproduct E is called *disjoint* if every coproduct inclusion $i_\alpha\colon E_\alpha \to E$ is a monomorphism and, whenever $\alpha \neq \beta$, $E_\alpha \times_E E_\beta$ is the initial object in $\mathcal{E}$.

The coproduct E is called *stable* (under pullback) if for every $f\colon D \to E$ in $\mathcal{E}$, the morphisms j_α obtained from the pullback diagrams

$$\begin{tikzcd} D \times_E E_\alpha \arrow[r] \arrow[d, "j_\alpha"'] & E_\alpha \arrow[d, "i_\alpha"] \\ D \arrow[r, "f"'] & E \end{tikzcd}$$

induce an isomorphism $\coprod_\alpha (D \times_E E_\alpha) \cong D$.

Remark 2.1.8. If every coproduct in $\mathcal{E}$ is stable, then the pullback operation $- \times_E D$ "commutes" with coproducts in the sense that $(\coprod_\alpha B_\alpha) \times_E D \cong \coprod_\alpha (B_\alpha \times_E D)$.

Remark 2.1.9. If a category $\mathcal{C}$ with an initial object $\emptyset$ has stable coproducts, then the existance of an arrow $X \to \emptyset$ implies $X \cong \emptyset$. Indeed, consider $\mathcal{C}(X, Z)$, which has at least one element since it contains the composition $X \to \emptyset \to Z$. We will prove that any two elements $f, g \in \mathcal{C}(X, Z)$ are equal.

By Remark 2.1.8, $X \cong X \times_\emptyset \emptyset \cong X \times_\emptyset (\emptyset \amalg \emptyset) \cong (X \times_\emptyset \emptyset) \amalg (X \times_\emptyset \emptyset) \cong X \amalg X$. Let ϕ represent this isomorphism $X \amalg X \to X$. Let i_0 and i_1 be the two natural maps $X \to X \amalg X$. Then $id_X = \phi i_0$ and $id_X = \phi i_1$. But ϕ is an isomorphism and so $i_0 = i_1$.

Now use f and g to induce the arrow $f \amalg g\colon X \amalg X \to Z$, i.e. $(f \amalg g) i_0 = f$ and $(f \amalg g) i_1 = g$. Since $i_0 = i_1$, then $f = g$.

Lemma 2.1.10. Let $\mathcal{C}$ be a category with disjoint and stable coproducts, and an initial object. Suppose $f_\alpha\colon A_\alpha \to B_\alpha$ are effective epimorphisms for all $\alpha \in \mathcal{A}$. Then $\coprod_\mathcal{A} f_\alpha\colon \coprod_\mathcal{A} A_\alpha \to \coprod_\mathcal{A} B_\alpha$ is an effective epimorphism (provided all necessary coproducts exist). Moreover, if $\mathcal{C}$ has all pullbacks and coproducts, and the f_α are universal effective epimorphisms, then $\coprod_\mathcal{A} f_\alpha$ is also a universal effective epimorphism.

Proof. Our basic argument is

$$\begin{aligned}
\coprod_{\alpha\in\mathcal{A}} B_\alpha &\cong \coprod_{\alpha\in\mathcal{A}} \mathrm{Coeq}\left(\begin{array}{c} A_\alpha \times_{B_\alpha} A_\alpha \\ \downarrow\downarrow \\ A_\alpha \end{array}\right) \\
&\cong \mathrm{Coeq}\left(\begin{array}{c} \coprod_{\alpha\in\mathcal{A}} (A_\alpha \times_{B_\alpha} A_\alpha) \\ \downarrow\downarrow \\ \coprod_{\alpha\in\mathcal{A}} A_\alpha \end{array}\right) \\
&\cong \mathrm{Coeq}\left(\begin{array}{c} \left(\coprod_{\alpha\in\mathcal{A}} A_\alpha\right) \times_{\coprod_{\beta\in\mathcal{A}} B_\beta} \left(\coprod_{\gamma\in\mathcal{A}} A_\gamma\right) \\ \downarrow\downarrow \\ \coprod_{\eta\in\mathcal{A}} A_\eta \end{array}\right)
\end{aligned}$$

The first isomorphism comes from assuming the f_α are effective epimorphisms. The second isomorphism comes from commuting colimits. The last isomorphism comes from the isomorphism

$$\coprod_{\alpha\in\mathcal{A}} (A_\alpha \times_{B_\alpha} A_\alpha) \cong \left(\coprod_{\alpha\in\mathcal{A}} A_\alpha\right) \times_{\coprod_{\beta\in\mathcal{A}} B_\beta} \left(\coprod_{\gamma\in\mathcal{A}} A_\gamma\right) \tag{2.1}$$

which we will now justify.

Let $B = \coprod_{\beta\in\mathcal{A}} B_\beta$. Since we know $\coprod_{\alpha\in\mathcal{A}} (A_\alpha \times_{B_\alpha} A_\alpha)$ exists, we will start here. First we will show that $A_\alpha \times_{B_\alpha} A_\alpha \cong A_\alpha \times_B A_\alpha$ by showing that the object $A_\alpha \times_{B_\alpha} A_\alpha$, which we know exists, satisfies the requirements of $\lim(A_\alpha \rightrightarrows B)$, which we have not assumed exists. Notice that our maps $A_\alpha \xrightarrow{\sigma_\alpha} B$ factor as $A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{i_\alpha} B$, where the i_α's are the canonical inclusion maps. This implies $A_\alpha \times_{B_\alpha} A_\alpha$ maps to the diagram $(A_\alpha \rightrightarrows B)$ appropriately. Now consider the parallel arrows $g, h\colon D \to A_\alpha$ such that $\sigma_\alpha g = \sigma_\alpha h$. By the factorization, $i_\alpha f_\alpha g = i_\alpha f_\alpha h$. Since i_α is a monomorphism, then

$f_\alpha g = f_\alpha h$. Now the universal property of the pullback $A_\alpha \times_{B_\alpha} A_\alpha$ gives us a unique map $D \to A_\alpha \times_{B_\alpha} A_\alpha$ that factors both g and h as desired. Hence $A_\alpha \times_{B_\alpha} A_\alpha$ is $\lim(A_\alpha \rightrightarrows B)$. Therefore $\coprod_{\alpha\in\mathcal{A}} (A_\alpha \times_{B_\alpha} A_\alpha) \cong \coprod_{\alpha\in\mathcal{A}} (A_\alpha \times_B A_\alpha)$.

Since coproducts are disjoint, then $B_\alpha \times_B B_\gamma = \emptyset$ whever $\alpha \neq \gamma$. Thus by Remark 2.1.9 and the following diagram

$$\begin{tikzcd}
A_\alpha \times_B A_\gamma \arrow[rr] \arrow[dd] \arrow[dr, dotted, "\exists"] & & A_\gamma \arrow[d, "f_\gamma"] \\
& B_\alpha \times_B B_\gamma \arrow[r] \arrow[d] & B_\gamma \arrow[d, "i_\gamma"] \\
A_\alpha \arrow[r, "f_\alpha"'] & B_\alpha \arrow[r, "i_\alpha"'] & B
\end{tikzcd}$$

we see that $A_\alpha \times_B A_\gamma = \emptyset$ whenever $\alpha \neq \gamma$. This implies that

$$\coprod_{\alpha\in\mathcal{A}} (A_\alpha \times_B A_\alpha) \cong \coprod_{\alpha,\gamma\in\mathcal{A}} (A_\alpha \times_B A_\gamma).$$

Lastly, the commutativity of coproducts and pullbacks (see Remark 2.1.8) yields

$$\coprod_{\alpha,\gamma\in\mathcal{A}} (A_\alpha \times_B A_\gamma) \cong \coprod_{\alpha\in\mathcal{A}} A_\alpha \times_B \coprod_{\gamma\in\mathcal{A}} A_\gamma$$

which completes the justification of (2.1).

We have now shown that $\coprod_{\mathcal{A}} f_\alpha$ is an effective epimorphism. The universality of $\coprod_{\mathcal{A}} f_\alpha$ is a consequence of the disjoint and stable coproducts. Indeed, suppose $\mathcal{C}$ has all pullbacks and let $D \to B$ be a given morphism. Stability of coproducts implies that $D \cong \amalg_{\alpha\in\mathcal{A}}(D \times_B B_\alpha)$. It follows that the following is a pullback square

$$\begin{tikzcd}
\coprod_{\alpha\in\mathcal{A}}(D \times_B B_\alpha \times_{B_\alpha} A_\alpha) \arrow[rr] \arrow[d, "g"'] & & \coprod_{\alpha\in\mathcal{A}} A_\alpha \arrow[d, "\amalg f_\alpha"] \\
\coprod_{\alpha\in\mathcal{A}} D \times_B B_\alpha \arrow[r, equal] & D \arrow[r] & \coprod_{\alpha\in\mathcal{A}} B_\alpha
\end{tikzcd}$$

where $g = \coprod_{\alpha \in \mathcal{A}} g_\alpha$ and $g_\alpha \colon D \times_B B_\alpha \times_{B_\alpha} A_\alpha \to D \times_B B_\alpha$ is the natural map. Moreover, g_α is the pullback of the universal effective epimorphism f_α. Thus each g_α is an effective epimorphism and so we have already shown that $\amalg_\alpha g_\alpha = g$ is a an effective epimorphism. $\square$

2.2 Index-Functor Category

In this section we define and discuss a special 2-category that will serve as a key tool in the proof of Theorem 3.1.1 and our manipulation of colimits. Additionally, we will prove a few observations and technical results that will be useful later.

For a fixed category $\mathcal{C}$, define $\mathscr{A}_\mathcal{C}$ to be the following 2-category:

- An object is a pair (I, F) where I is a small category and $F \colon I \to \mathcal{C}$ is a functor.
- A morphism is a pair $(g, \eta) \colon (I, F) \to (I', F')$. The g is a functor $g \colon I \to I'$. The η is a natural transformation $\eta \colon F \to F' \circ g$. Morally, we think of g as almost being an arrow in $(Cat \downarrow \mathcal{C})$ where Cat is the category of small categories; the natural transformation η replaces the commutativity required for an arrow in the overcategory.
- A 2-morphism from $(f, \eta_f) \colon (I, D) \to (J, E)$ to $(g, \eta_g) \colon (I, D) \to (J, E)$ is a natural transformation $\theta \colon f \to g$ such that for each i in the objects of I, the following is a commutative diagram

$$\begin{tikzcd} & Di \arrow[dl, "(\eta_f)_i"'] \arrow[dr, "(\eta_g)_i"] & \\ Efi \arrow[rr, "E\theta i"'] & & Egi. \end{tikzcd}$$

Definition 2.2.1. We call $\mathscr{A}_\mathcal{C}$ the *Index-Functor Category* for $\mathcal{C}$.

Notation 2.2.2. Let $*$ be the category consisting of one object and no non-identity morphisms. We will abuse notation and also use $*$ to represent its unique object.

Notation 2.2.3. For any object Z of $\mathcal{C}$, let cZ be the object of $\mathscr{A}_{\mathcal{C}}$ given by $(*, c_Z)$ where $c_Z(*) = Z$, i.e. cZ is the constant diagram on Z.

Notation 2.2.4. For a sieve T on X, we will use T as shorthand notation for the object (T, U) of $\mathscr{A}_{\mathcal{C}}$. (See Notation 1.0.1 for the definition of U.)

Notation 2.2.5. Let T be a sieve on X. We have a canonical map $\phi_T \colon T \to cX$ given by $\phi_T = (t, \varphi_T)$ where t is the terminal map $T \to *$ and $\varphi_T \colon U \to (c_X \circ t)$ is given by $(\varphi_T)_f = f$ for $f \in T$.

Remark 2.2.6. Notice that for all objects V and W of $\mathcal{C}$,

$$\mathscr{A}_{\mathcal{C}}(cV, cW) \cong \mathcal{C}(V, W)$$

since the only non-determined information in a map from cV to cW is the natural transformation $c_V \to c_W \circ t$, which is just a map $V \to W$ in $\mathcal{C}$.

Next we include a few observations:

Lemma 2.2.7. If $D \colon I \to \mathcal{C}$ and X is a cocone for D, then we have a morphism $\phi \colon (I, D) \to cX$ in $\mathscr{A}_{\mathcal{C}}$. The induced morphism $\phi^* \colon \mathscr{A}_{\mathcal{C}}(cX, cY) \to \mathscr{A}_{\mathcal{C}}((I, D), cY)$ is a bijection for all objects Y of $\mathcal{C}$ if and only if X is a colimit for D.

Proof. Left to the reader. □

Lemma 2.2.8. Let $(f, \eta_f), (g, \eta_g) \colon (I, F) \to (J, G)$ be two morphisms in $\mathscr{A}_{\mathcal{C}}$. If there exists a 2-morphism $\alpha \colon (f, \eta_f) \to (g, \eta_g)$, then the induced maps
$(f, \eta_f)^*, (g, \eta_g)^* \colon \mathscr{A}_{\mathcal{C}}((J, G), cY) \to \mathscr{A}_{\mathcal{C}}((I, F), cY)$ are equal for all objects Y in $\mathcal{C}$.

Proof. Let $(k, \eta_k) \in \mathscr{A}_{\mathcal{C}}((J, G), cY)$. Then

$$(f, \eta_f)^*(k, \eta_k) = (k \circ f, f^*(\eta_k) \circ \eta_f) \quad \text{and} \quad (g, \eta_g)^*(k, \eta_k) = (k \circ g, g^*(\eta_k) \circ \eta_g).$$

But k must be the terminal functor $J \to *$ and thus $k \circ f = k \circ g$. To see that $f^*(\eta_k) \circ \eta_f = g^*(\eta_k) \circ \eta_g$ fix an object $i \in I$ and notice that we have the following diagram:

$$\begin{tikzcd}
 & F_i \arrow[dl, "(\eta_f)_i"'] \arrow[dr, "(\eta_g)_i"] & \\
Gfi \arrow[rr, "G\alpha_i"'] \arrow[dr, "\eta_k"'] & & Ggi \arrow[dl, "\eta_k"] \\
 & Y &
\end{tikzcd}$$

where the upper part of the diagram commutes because α is a 2-morphism and the lower part commutes because of the natural transformation η_k. Since the left vertical composition in the above diagram is $(f^*(\eta_k) \circ \eta_f)_i$ and the right vertical composition is $(g^*(\eta_k) \circ \eta_g)_i$, then this completes the proof. $\square$

Before the last result we include a reminder about Grothendieck constructions. Whenever we have a functor $G \colon A \to Cat$, where Cat is the category of small categories, we can create a *Grothendieck construction* of G, which we will denote $\mathrm{Gr}(G)$. The objects of $\mathrm{Gr}(G)$ are pairs (a, τ) where a is an object of A and τ is an object of $G(a)$. The morphisms are pairs $(f, g) \colon (a, \tau) \to (a', \tau')$ where $f \colon a \to a'$ is a morphism in A and $g \colon Gf(\tau) \to \tau'$ is a morphism in $G(a')$.

Proposition 2.2.9. Let A and $\mathcal{C}$ be categories. Suppose there exists functors $G \colon A \to Cat$, $\theta \colon A \to \mathcal{C}$ and $\sigma \colon \mathrm{Gr}(G) \to \mathcal{C}$, and a morphism in $\mathscr{A}_{\mathcal{C}}$ of the form $F = (f, \eta) \colon (\mathrm{Gr}(G), \sigma) \to (A, \theta)$ where $f(a, \tau) = a$. If for all objects a of A, $\theta(a)$ is the colimit of $\sigma(a, -) \colon G(a) \to \mathcal{C}$ where the isomorphism is induced by η, then the induced map $F^*\colon \mathscr{A}_{\mathcal{C}}((A,\theta),cY) \to \mathscr{A}_{\mathcal{C}}((\mathrm{Gr}(G),\sigma),cY)$ is a bijection for all objects Y of $\mathcal{C}$.

Remark: Fix $a \in A$, then $\eta_{(a,-)}\colon \sigma(a,-) \to \theta(a)$ is a natural transformation. In particular, $(\theta(a), \eta_{(a,-)})$ is a cocone under $\sigma(a,-)$. Our colimit assumption is specifically that this cocone is universal.

Proof. We start by showing that F^* is an injection; let $(k,\chi_k),(l,\chi_l) \in \mathscr{A}_{\mathcal{C}}((A,\theta),cY)$ such that $F^*(k,\chi_k) = F^*(l,\chi_l)$, i.e. $(k\circ f, f^*(\chi_k)\circ\eta) = (l\circ f, f^*(\chi_l)\circ\eta)$. Since both k and l are functors $A \to *$ then they are both the terminal map, which is unique and hence $k = l$.

Now fix $a \in A$. Consider $(a,\tau) \in \mathrm{Gr}(G)$. For both $t = k$ and $t = l$, the natural transformations (i.e. second coordinates of the maps in question) at (a,τ) take the form

$$(f^*(\chi_t)\circ\eta)_{(a,\tau)} = (\chi_t)_a \circ \eta_{(a,\tau)}\colon \sigma(a,\tau) \to \theta f(a,\tau) = \theta(a) \to c_Y t(a) = Y$$

where c_Y comes from $cY = (*, c_Y)$. Moreover, since η and χ_t are both natural transformations, then these maps $\sigma(a,\tau) \to Y$ are compatible among all arrows in $G(a)$. But by assumption $\varinjlim_{G(a)} \sigma(a,-) \cong \theta(a)$. Thus the maps $(\chi_t)_a \circ \eta_{(a,\tau)}$ define a map from the colimit, i.e. from $\theta(a)$ to Y. By the universal property of colimits, there is only one choice for this map, namely $(\chi_t)_a$. Moreover, since $(\chi_k)_a\circ\eta_{(a,\tau)} = (\chi_l)_a\circ\eta_{(a,\tau)}$, then $(\chi_k)_a$ and $(\chi_l)_a$ must define the same map out of the colimit. Therefore $(\chi_k)_a = (\chi_l)_a$ for all $a \in A$ and this finishes the proof of injectivity.

To prove surjectivity, let $(m,\chi_m) \in \mathscr{A}_{\mathcal{C}}((\mathrm{Gr}(G),\sigma),cY)$. Let (k,χ_k) be the following pair:

- $k\colon A \to *$ is the terminal functor

– χ_k is a collection of maps, one for each object a of A, from $\theta(a)$ to Y. The map for object a is induced by the maps $(\chi_m)_{(a,\tau)} \colon \sigma(a,\tau) \to Y$ for all τ in $G(a)$. Note that these maps exist and are well defined because χ_m is a natural transformation and $\varinjlim_{G(a)} \sigma(a,-) \cong \theta(a)$.

We claim two things: $(k,\chi_k) \in \mathscr{A}_{\mathcal{C}}((A,\theta), cY)$ and $F^*(k,\chi_k) = (m,\chi_m)$

To prove the first claim we merely need to show that χ_k is a natural transformation $\theta \to c_Y \circ k$. By its definition, it is clear that χ_k does the correct thing on objects; all we need to check is what it does to arrows in A. Specifically, let $g \colon a \to b$ be a morphism in A. Then for any $\tau \in G(a)$, $(g, id_{Gg(\tau)})$ is a morphism in $\mathrm{Gr}(G)$. Since $\chi_m \colon \sigma \to c_Y \circ m$ is a natural transformation, then we have the following commutative diagram

$$\begin{tikzcd}
\sigma(a,\tau) \arrow[r, "\sigma(g,id)"] \arrow[d, "\chi_m"'] & \sigma(b, Gg(\tau)) \arrow[d, "\chi_m"] \\
Y \arrow[r, "id"'] & Y
\end{tikzcd}$$

and in particular, the map from diagram $\sigma(a,-) \colon G(a) \to \mathcal{C}$ to Y factors through the map from diagram $\sigma(b,-) \colon G(b) \to \mathcal{C}$ to Y. Thus the induced map $(\chi_k)_a \colon \varinjlim_{G(a)} \sigma(a,-) \to Y$ factors through $(\chi_k)_b$. Furthermore, the natural transformation $\eta \colon \sigma \to \theta f$, which induces $\varinjlim_{G(c)} \sigma(c,-) \cong \theta(c)$, ensures that this factorization is $(\chi_k)_a = (\chi_k)_b \circ \theta(g)$, which completes the proof that χ_k is a natural transformation.

To prove the second claim, we need to show that $F^*(k,\chi_k) = (k \circ f, f^*(\chi_k) \circ \eta)$ equals (m,χ_m). Since both k and m are terminal functors, then $k \circ f = m$. To see that $f^*(\chi_k) \circ \eta = \chi_m$, fix an object $(a,\tau) \in \mathrm{Gr}(G)$. Notice that $(f^*(\chi_k) \circ \eta)_{(a,\tau)}$ equals $(\chi_k)_a \circ \eta_{(a,\tau)}$, which is the composition

$$
\begin{tikzcd}
\sigma(a,\tau) \arrow[r] \arrow[rr, bend right, "\eta"'] & \operatorname{colim}_{G(a)} \sigma(a,-) \arrow[r, "\text{induced by } \eta", "\cong"'] & \theta(a) \arrow[r, "\chi_k"] & Y.
\end{tikzcd}
$$

But χ_k was created by inducing maps from the colimit to Y based on χ_m, which means that this composition must also be $(\chi_m)_{(a,\tau)}$. Therefore, $f^*(\chi_k) \circ \eta = \chi_m$ and our second claim has been proven, which finishes the proof. $\square$

2.3 Generalized Sieves

In this section we define and discuss a particular generalization for a sieve; this will be a key tool in the proofs of Theorems 3.1.1 and 8.0.2 (where we show that certain collections form Grothendieck topologies). Additionally, we define two special functors.

Definition 2.3.1. Fix a positive integer n. Let $T_1,\ T_2, \dots,\ T_n$ be sieves on X. A *generalized sieve*, denoted by ${}_X[T_1T_2 \dots T_n]$, is the following category:

– objects $(\rho_1,\ \rho_2, \dots,\ \rho_n)$ are n-tuples of arrows in $\mathcal{C}$ such that the composition $\rho_1 \circ \rho_2 \circ \dots \circ \rho_i \in T_i$ for all $i = 1, \dots, n$. Pictorially we can visualize this as

$$
\begin{tikzcd}
X & A_1 \arrow[l, "\rho_1", "\in T_1"'] & A_2 \arrow[l, "\rho_2"] \arrow[ll, bend right, "\in T_2"'] & \dots \arrow[l, "\rho_3"] & A_n. \arrow[l, "\rho_n"] \arrow[llll, bend left, "\in T_n"]
\end{tikzcd}
$$

– morphisms $(f_1,\ f_2, \dots,\ f_n)$ from $(\rho_1,\ \rho_2, \dots,\ \rho_n)$ to $(\tau_1,\ \tau_2, \dots,\ \tau_n)$ are n-tuples of arrows in $\mathcal{C}$ where $f_i \colon \mathrm{domain}(\rho_i) \to \mathrm{domain}(\tau_i)$ such that all squares in the following diagram commute

$$
\begin{tikzcd}
X \arrow[d, "id"'] & A_1 \arrow[l, "\rho_1"'] \arrow[d, "f_1"'] & A_2 \arrow[l, "\rho_2"'] \arrow[d, "f_2"'] & \dots \arrow[l, "\rho_3"'] & A_n \arrow[l, "\rho_n"'] \arrow[d, "f_n"'] \\
X & B_1 \arrow[l, "\tau_1"] & B_2 \arrow[l, "\tau_2"] & \dots \arrow[l, "\tau_3"] & B_n. \arrow[l, "\tau_n"]
\end{tikzcd}
$$

For example, if T is a sieve on X, then ${}_X[T]$ is T (as categories).

Remark 2.3.2. For sieves $T_1, \dots, T_n$ on X we can define a functor

$$G\colon {}_X[T_1T_2\dots T_{n-1}] \to Cat, \quad (\rho_1, \dots, \rho_{n-1}) \mapsto (\rho_1 \circ \cdots \circ \rho_{n-1})^* T_n.$$

Then the Grothendieck construction for G is ${}_X[T_1T_2\dots T_n]$. Indeed, this is easy to see once we view the objects of ${}_X[T_1T_2\dots T_n]$ as pairs

$$((\rho_1, \dots, \rho_{n-1}) \in {}_X[T_1 \dots T_{n-1}], \tau \in G(\rho_1, \dots, \rho_{n-1})).$$

Like a sieve, a generalized sieve ${}_X[T_1 \dots T_n]$ can be viewed as a subcategory of $(\mathcal{C} \downarrow X)$. Thus we will use U (see Notation 1.0.1) as the functor ${}_X[T_1T_2\dots T_n] \to \mathcal{C}$ given by $(\rho_1, \rho_2, \dots, \rho_n) \mapsto$ domain ρ_n. Note: for any morphism $(f_1,\ f_2, \dots,\ f_n)$, $U(f_1,\ f_2, \dots,\ f_n) = f_n$.

Definition 2.3.3. Let $T_1,\ T_2, \dots,\ T_n$ be sieves on X (with $n \geq 2$), we define a 'forgetful functor'

$$\mathscr{F}\colon {}_X[T_1T_2\dots T_n] \to {}_X[T_1T_2\dots T_{n-1}], \quad (\rho_1, \rho_2, \dots, \rho_n) \mapsto (\rho_1, \rho_2, \dots, \rho_{n-1}).$$

Pictorially,

$$X \xleftarrow{\rho_1} A_1 \xleftarrow{\rho_2} A_2 \xleftarrow{\rho_3} \dots \xleftarrow{\rho_{n-1}} A_{n-1} \xleftarrow{\rho_n} A_n$$

$$\overset{\mathscr{F}}{\Longmapsto} X \xleftarrow{\rho_1} A_1 \xleftarrow{\rho_2} A_2 \xleftarrow{\rho_3} \dots \xleftarrow{\rho_{n-1}} A_{n-1}.$$

Remark 2.3.4. Actually, the above definition only needs $n \geq 1$. In the $n = 1$ case, our forgetful functor is $\mathscr{F}\colon {}_X[T_1] \to {}_X[\,]$, where ${}_X[\,]$ is the category with unique object $(id_X\colon X \to X)$ and no non-identity morphisms, and is defined by $\rho \mapsto id_X$.

Now we take this functor $\mathscr{F}$ and use it to make an arrow in $\mathscr{A}_{\mathfrak{C}}$:

Definition 2.3.5. For any sieves $T_1,\ T_2, \dots,\ T_n$ on X (with $n \geq 2$), define a map in $\mathscr{A}_{\mathfrak{C}}$ called $\widetilde{\mathscr{F}}\colon ({}_X[T_1T_2 \dots T_n], U) \to ({}_X[T_1T_2 \dots T_{n-1}], U)$ by $\widetilde{\mathscr{F}} = (\mathscr{F}, \eta_{\mathscr{F}})$ where $\eta_{\mathscr{F}}\colon U \to (U \circ \mathscr{F})$ is given by $(\eta_{\mathscr{F}})_{(\rho_1,\rho_2,\dots,\rho_n)} = \rho_n$.

The fact that $\eta_{\mathscr{F}}$ is a natural transformation can be seen easily from the pictorial view of morphisms. Specifically, consider the morphism $(f_1, f_2, \dots, f_n)$; this morphism gives us a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "id"'] & A_1 \arrow[l, "\rho_1"'] \arrow[d, "f_1"'] & A_2 \arrow[l, "\rho_2"'] \arrow[d, "f_2"'] & \dots \arrow[l, "\rho_3"'] & A_{n-1} \arrow[l, "\rho_{n-1}"'] \arrow[d, "f_{n-1}"'] & A_n \arrow[l, "\rho_n"'] \arrow[d, "f_n"'] \\
X & B_1 \arrow[l, "\tau_1"] & B_2 \arrow[l, "\tau_2"] & \dots \arrow[l, "\tau_3"] & B_{n-1} \arrow[l, "\tau_{n-1}"'] & B_n \arrow[l, "\tau_n"]
\end{tikzcd}$$

but the rightmost commutative square of the above diagram can be relabelled to give us the following commutative diagram

$$\begin{tikzcd}
U \circ \mathscr{F}(\rho_1, \rho_2, \dots, \rho_n) \arrow[d, "U\circ\mathscr{F}(f_1,f_2,\dots,f_n)"'] & U(\rho_1, \rho_2, \dots, \rho_n) \arrow[l, "\eta_{\mathscr{F}}"'] \arrow[d, "U(f_1,f_2,\dots,f_n)"'] \\
U \circ \mathscr{F}(\tau_1, \tau_2, \dots, \tau_n) & U(\tau_1, \tau_2, \dots, \tau_n) \arrow[l, "\eta_{\mathscr{F}}"]
\end{tikzcd}$$

and it is this diagram that shows $\eta_{\mathscr{F}}$ is a natural transformation.

Definition 2.3.6. Let $T_1,\ T_2, \dots,\ T_n$ be sieves on X (with $n \geq 2$), we define a 'composition functor'

$$\mu\colon\ {}_X[T_1T_2 \dots T_n] \to {}_X[T_2 \dots T_n], \quad (\rho_1, \rho_2, \dots, \rho_n) \mapsto (\rho_1 \circ \rho_2, \rho_3, \dots, \rho_n).$$

Pictorially,

$$X \xleftarrow{\rho_1} A_1 \xleftarrow{\rho_2} A_2 \xleftarrow{\rho_3} \dots \xleftarrow{\rho_{n-1}} A_{n-1} \xleftarrow{\rho_n} A_n$$

$$\stackrel{\mu}{\Longmapsto} X \xleftarrow{\rho_1\circ\rho_2} A_2 \xleftarrow{\rho_3} A_3 \xleftarrow{\rho_4} \dots \xleftarrow{\rho_n} A_n.$$

Now we take this functor μ and use it to make an arrow in $\mathscr{A}_{\mathcal{C}}$:

Definition 2.3.7. For any sieves $T_1, T_2, \ldots, T_n$ on X (with $n \geq 2$). Define $\widetilde{\mu}\colon ({}_X[T_1T_2\ldots T_n], U) \to ({}_X[T_2T_3\ldots T_n], U)$ by $\widetilde{\mu} = (\mu, \eta_\mu)$ where $\eta_\mu\colon U \to (U \circ \mu)$ is given by $(\eta_\mu)_{(\rho_1,\rho_2,\ldots,\rho_n)} = id_{\text{domain } \rho_n}$.

Lastly, we include an easy corollary.

Corollary 2.3.8. Let V and W be sieves on X such that for all $f \in V$, f^*W is a colim sieve. Fix an integer $n \geq 0$ and let $T_1, T_2, \ldots, T_n$ be a list of sieves on X (note: $n = 0$ corresponds to the empty list). Then the induced map $\widetilde{\mathscr{F}}^*\colon \mathscr{A}_{\mathcal{C}}({}_X[T_1T_2\ldots T_nV], cY) \to \mathscr{A}_{\mathcal{C}}({}_X[T_1T_2\ldots T_nVW], cY)$ is a bijection for all objects Y of $\mathcal{C}$.

Proof. This is an immediate application of Proposition 2.2.9 and Remark 2.3.2.

$\square$

CHAPTER III

UNIVERSAL COLIM SIEVES AND THE CANONICAL TOPOLOGY

In this section we show that the collection of all universal colim sieves forms the canonical topology; this folklore result is mentioned in [6], but we give a new proof that generalizes to a homotopical setting, which we consider later. First we show that this collection forms a Grothendieck topology. Second we show that the two topologies are the same.

3.1 Universal Colim Sieves form a Topology

This section is dedicated to proving the following result:

Theorem 3.1.1. Let $\mathcal{C}$ be any category. The collection of all universal colim sieves on $\mathcal{C}$ forms a Grothendieck topology.

Let $\mathcal{U}$ be the collection of universal colim sieves for the category $\mathcal{C}$ with $\mathcal{U}(X)$ the collection of universal colim sieves on X. A topology must have three properties:

1. contain all of the maximal sieves,
2. satisfy the stability axiom
3. satisfy the transitivity axiom

Note: our terminology follows Maclane's and Moerdijk's in [10].

The first two properties, i.e. the maximal and stability axioms, are easy to check. Indeed, stability is immediate from the definition of universal colim sieve whereas the maximal sieve on X is the category $(\mathcal{C} \downarrow X)$, which has a terminal object, namely

$id\colon X \to X$. Thus the inclusion functor $L\colon * \to (\mathcal{C} \downarrow X)$ given by $L(*) = id$ (see Notation 2.2.2) is a final functor. Hence by [9, Theorem 1, Section 3, Chapter IX]

$$\underset{(\mathcal{C}\downarrow X)}{\underrightarrow{\mathrm{colim}}}\, U \cong \underset{*}{\underrightarrow{\mathrm{colim}}}\, UL \cong UL(*) = X$$

and so the maximal sieve on X is a colim sieve. Moreover, for all $f\colon Y \to X$ in $\mathcal{C}$, $f^*(\mathcal{C} \downarrow X) = (\mathcal{C} \downarrow Y)$, which by the previous argument is a colim sieve on Y. Therefore, $(\mathcal{C} \downarrow X) \in \mathcal{U}(X)$.

Proving that the the transitivity axiom holds will require more than a few paragraphs and will be the remainder of this section. From here on out, we fix $S \in \mathcal{U}(X)$ and a sieve R on X such that for all $f \in S$, $f^*R \in \mathcal{U}(\text{domain } f)$. We need to prove that $R \in \mathcal{U}(X)$.

Remark 3.1.2. Throughout this proof we will be using notation and results from Sections 2.2 and 2.3.

I. Reduction

First we will remove the need to show universality. Indeed, up to notation, for any morphism α in $\mathcal{C}$ with codomain X, we have the same assumptions for α^*R as we have for R (when we use α^*S instead of S). In particular, this means that showing R is a colim sieve on X will also show (up to notation) that each α^*R is a colim sieve. Therefore it suffices to show that R is a colim sieve.

Second we will rephrase what it means for R to be a colim sieve. By definition, R is a colim sieve if and only if X is a colimit for R. Now Lemma 2.2.7 tells us this is equivalent to the induced map $\phi_R^*\colon \mathscr{A}_{\mathcal{C}}(cX, cY) \to \mathscr{A}_{\mathcal{C}}(R, cY)$ being a bijection for all objects Y of $\mathcal{C}$ (see Notation 2.2.5 for the definition of ϕ_R).

Reduction: To prove that R is a universal colim sieve, it suffices to prove that $\phi_R^*\colon \mathscr{A}_{\mathcal{C}}(cX, cY) \to \mathscr{A}_{\mathcal{C}}(R, cY)$ is a bijection for all objects Y of $\mathcal{C}$.

II. Outline

We will be using the following noncommutative diagram in $\mathscr{A}_{\mathcal{C}}$:

$$\begin{tikzcd}
R \arrow[r, "\phi_R"] & cX \\
{}_X[RS] \arrow[u, "\widetilde{\mathscr{F}}"] & S \arrow[u, "\phi_S"'] \\
{}_X[RSR] \arrow[u, "\widetilde{\mathscr{F}}"] \arrow[r, "\widetilde{\mu}"'] & {}_X[SR]. \arrow[u, "\widetilde{\mathscr{F}}"'] \arrow[uul, "\widetilde{\mu}"]
\end{tikzcd} \tag{3.1}$$

Note: ${}_X[T_1T_2\dots T_n]$ is shorthand for $({}_X[T_1T_2\dots T_n], U)$, just like how R and S are shorthand for (R, U) and (S, U) respectively.

- We will show that the upper right triangle commutes and the lower left triangle commutes up to a 2-morphism.
- Then we will apply $\mathscr{A}_{\mathcal{C}}(-, cY)$ levelwise to the diagram, which will result in a commutative diagram by Lemma 2.2.8.
- It will then follow formally that the induced ϕ_R^* is a bijection. By the reduction, this will complete the proof of transitivity.

III. Proof of Transitivity

Lemma 3.1.3. In diagram (3.1), the upper right triangle commutes.

Proof. We start by unpacking what the compositions in the diagram are:

$$\phi_R \circ \widetilde{\mu} = (t, \varphi_R) \circ (\mu, \eta_\mu) = (t \circ \mu, \mu^* \varphi_R \circ \eta_\mu)$$

$$\phi_S \circ \widetilde{\mathscr{F}} = (t, \varphi_S) \circ (\mathscr{F}, \eta_{\mathscr{F}}) = (t \circ \mathscr{F}, \mathscr{F}^* \varphi_S \circ \eta_{\mathscr{F}})$$

Since t is the terminal map, then $t \circ \mu = t \circ \mathscr{F}$. To see that the natural transformations are the same fix $(\rho, \tau) \in {}_X[SR]$. Then

$$(\mu^* \varphi_R \circ \eta_\mu)_{(\rho,\tau)} = (\varphi_R)_{\mu(\rho,\tau)} \circ id = (\varphi_R)_{\rho \circ \tau} = \rho \circ \tau$$

and

$$(\mathscr{F}^* \varphi_S \circ \eta_{\mathscr{F}})_{(\rho,\tau)} = (\varphi_S)_{\mathscr{F}(\rho,\tau)} \circ \tau = (\varphi_S)_\rho \circ \tau = \rho \circ \tau.$$

Since the natural transformations are the same on all objects, the proof is complete.

□

At this point it would be nice if the lower left triangle in the diagram also commuted, however, it does not. Instead, it contains a 2-morphism:

Lemma 3.1.4. There exists a 2-morphism $\theta\colon \widetilde{\mu} \circ \widetilde{\mu} \to \widetilde{\mathscr{F}} \circ \widetilde{\mathscr{F}}$ where $\widetilde{\mu} \circ \widetilde{\mu}, \widetilde{\mathscr{F}} \circ \widetilde{\mathscr{F}}\colon {}_X[RSR] \to R$.

Two remarks: First, ${}_X[R] = R$. Second, this lemma and (a similar) proof hold for ${}_X[T_1 T_2 \dots T_n] \to {}_X[T_1 T_2 \dots T_{n-2}]$ when all $T_{odd} = T_1$ and $T_{even} = T_2$. The two morphisms "are" $\mu \circ \mu\colon (\rho_1, \dots, \rho_n) \mapsto (\rho_1 \circ \rho_2 \circ \rho_3, \rho_4, \dots, \rho_n)$ and $\mathscr{F} \circ \mathscr{F}\colon (\rho_1, \dots, \rho_n) \mapsto (\rho_1, \dots, \rho_{n-2})$.

Proof. We start by recalling $\mu \circ \mu \colon \left[X \xleftarrow{\rho} A \xleftarrow{\tau} B \xleftarrow{\gamma} C\right] \mapsto \left[X \xleftarrow{\rho\tau\gamma} C\right]$ and $\mathscr{F} \circ \mathscr{F} \colon \left[X \xleftarrow{\rho} A \xleftarrow{\tau} B \xleftarrow{\gamma} C\right] \mapsto \left[X \xleftarrow{\rho} A\right]$. Now define $\theta \colon \widetilde{\mu} \circ \widetilde{\mu} \to \widetilde{\mathscr{F}} \circ \widetilde{\mathscr{F}}$ by $(\theta)_{(\rho,\tau,\gamma)} = \tau \circ \gamma$. We claim that this θ is the desired 2-morphism.

First, θ is clearly a natural transformation from μ^2 to $\mathscr{F}^2$. Indeed, consider the following object in ${}_X[RSR]$:

$$\begin{tikzcd}
X & A \arrow[l, "\in R", "\rho"'] & B \arrow[l, "\tau"] \arrow[ll, bend right, "\in S"'] & C. \arrow[l, "\gamma"] \arrow[lll, bend right, "\in R"']
\end{tikzcd}$$

Notice that θ does the correct thing on objects since $\mu^2(\rho, \tau, \gamma) = X \xleftarrow[\rho\circ\tau\circ\gamma]{\in R} C$ and $\mathscr{F}^2(\rho, \tau, \gamma) = X \xleftarrow[\rho]{\in R} A$, and thus $\theta_{(\rho,\tau,\gamma)} = \tau \circ \gamma \colon C \to A$ is a morphism from $\mu^2(\rho, \tau, \gamma)$ to $\mathscr{F}^2(\rho, \tau, \gamma)$ in R. It is similarly easy to see that θ behaves compatibly with the morphisms of ${}_X[RSR]$.

Second, fix $(\rho, \tau, \gamma) \in {}_X[RSR]$. We also need to know that the diagram

$$\begin{tikzcd}
& \mathrm{domain}(\gamma) \arrow[dl, "id"'] \arrow[dr, "\tau\circ\gamma"] & \\
\mathrm{domain}(\rho\circ\tau\circ\gamma) \arrow[rr, "\theta\,=\,\tau\circ\gamma"'] & & \mathrm{domain}(\rho)
\end{tikzcd}$$

is commutative, which it clearly is. Therefore, θ is our desired 2-morphism. $\square$

Now fix Y, an object of $\mathfrak{C}$, and apply $\mathscr{A}_{\mathfrak{C}}(-, cY)$ to diagram (3.1) in order to obtain the following diagram of sets:

$$\begin{tikzcd}
\mathscr{A}_{\mathfrak{C}}(R, cY) \arrow[d, "\widetilde{\mathscr{F}}^*"'] \arrow[ddr, "\widetilde{\mu}^*"] & \mathscr{A}_{\mathfrak{C}}(cX, cY) \arrow[l, "\phi_R^*"'] \arrow[d, "\phi_S^*"] \\
\mathscr{A}_{\mathfrak{C}}({}_X[RS], cY) \arrow[d, "\widetilde{\mathscr{F}}^*"'] & \mathscr{A}_{\mathfrak{C}}(S, cY) \arrow[d, "\widetilde{\mathscr{F}}^*"] \\
\mathscr{A}_{\mathfrak{C}}({}_X[RSR], cY) & \mathscr{A}_{\mathfrak{C}}({}_X[SR], cY). \arrow[l, "\widetilde{\mu}^*"]
\end{tikzcd} \tag{3.2}$$

We will use this diagram to prove that ϕ_R^* is a bijection.

The upper right triangle in diagram (3.2) commutes by Lemma 3.1.3. Moreover, since the lower left triangle in the first diagram contained a 2-morphism (by Lemma 3.1.4), then Lemma 2.2.8 shows that the lower left triangle in diagram (3.2) commutes. Thus (3.2) is a commutative diagram of sets.

Now we will discuss some of the morphisms in (3.2). First, notice that by Lemma 2.2.7, since S is a colim sieve, ϕ_S^* is a bijection. Second, notice that Corollary 2.3.8 implies that all of the maps $\widetilde{\mathscr{F}}^*$ in diagram (3.2) are bijections. Indeed, by Corollary 2.3.8, our assumptions on R imply that $\widetilde{\mathscr{F}}^*\colon \mathscr{A}_{\mathfrak{C}}(S, cY) \to \mathscr{A}_{\mathfrak{C}}({}_X[SR], cY)$ and $\widetilde{\mathscr{F}}^*\colon \mathscr{A}_{\mathfrak{C}}({}_X[RS], cY) \to \mathscr{A}_{\mathfrak{C}}({}_X[RSR], cY)$ are bijections, and our assumptions on S imply that $\widetilde{\mathscr{F}}^*\colon \mathscr{A}_{\mathfrak{C}}(R, cY) \to \mathscr{A}_{\mathfrak{C}}({}_X[RS], cY)$ is a bijection. Hence all vertical maps in diagram (3.2) are isomorphisms.

We summarize the results about diagram (3.2): we have commutative triangles that combine to make a commutative diagram of sets of the form

$$\begin{tikzcd} \mathscr{A}_{\mathfrak{C}}(R, cY) \arrow[d, "\cong"'] \arrow[dr, "\alpha"] & \mathscr{A}_{\mathfrak{C}}(cX, cY) \arrow[l, "\phi_R^*"'] \arrow[d, "\cong"] \\ A & B. \arrow[l] \end{tikzcd} \tag{3.3}$$

Notice that some of the details mentioned in diagram (3.2) are not mentioned in the above diagram. Indeed, we only need to know that for each Y some such A, B and α exist, their specific values are not required; diagram (3.2) is what guarantees their existance.

Using the lower left triangle in diagram (3.3) we see that α is an injection. Whereas the upper right triangle in diagram (3.3) shows that α is a surjection. Therefore, α is a bijection. Now the commutativity of the upper right triangle in diagram (3.3) implies that ϕ_R^* is a bijection. Hence we have completed the proof of transitivity.

3.2 The Canonical Topology

In this section we give an explicit presentation and basis for the canonical topology. We start by giving the presentation:

Theorem 3.2.1. For any (locally small) category $\mathcal{C}$, the collection of all universal colim sieves on $\mathcal{C}$ is the canonical topology.

Proof. We start with a fact that will be used a few times: The equalizer in the sheaf condition can be expressed as a limit over a covering sieve. Specifically, for a presheaf F and covering sieve S

$$\operatorname{Eq}\left(\prod_{A \xrightarrow{f} X \in S} F(A) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{\substack{B \xrightarrow{g} A \\ A \xrightarrow{f} X \in S}} F(B)\right) = \varprojlim_{S} FU \tag{3.4}$$

where the fg component of $\alpha((x_f)_{f\in S})$ is x_{fg} and of $\beta((x_f)_{f\in S})$ is $Fg(x_f)$ [see 9, Theorem 2, Section 2, Chapter V].

Let $\mathcal{U}$ be the universal colim sieve topology for the category $\mathcal{C}$ with $\mathcal{U}(X)$ the collection of universal colim sieves on X. In a similar vein, let C be the canonical topology for $\mathcal{C}$. Let rM denoted the representable presheaf on M, i.e. for all objects K of $\mathcal{C}$, $rM(K) = \mathcal{C}(K, M)$. We will show that the universal colim sieves form a "larger topology" than the canonical topology, i.e. $C(X) \subset \mathcal{U}(X)$ for all objects X, and that $\mathcal{U}$ is subcanonical, i.e. that $\mathcal{U}$ is a topology where all representable presheaves are sheaves. This will prove the desired result because the canonical topology is the largest subcanonical topology.

To see that $C(X) \subset \mathcal{U}(X)$, let $S \in C(X)$, $f\colon Y \to X$ be a morphism and M be an object in $\mathcal{C}$. Since $f^*S \in C(Y)$ and rM is a sheaf in the canonical topology, then it follows from the the sheaf condition and (3.4) that

$$rM(Y) \cong \varprojlim_{f^*S}(rM \circ U).$$

Thus by rewriting what $rM(-)$ means, we get

$$\mathcal{C}(Y, M) \cong \lim_{g \in f^*S} \mathcal{C}\left(U(g), M\right)$$

for every object M. This formally implies that $\underrightarrow{\mathrm{colim}}_{f^*S}\, U$ exists and

$$\mathcal{C}(Y, M) \cong \mathcal{C}\left(\underrightarrow{\mathrm{colim}}_{f^*S}\, U, M\right)$$

for all objects M of $\mathcal{C}$. Now by Yoneda's Lemma, $Y \cong \underrightarrow{\mathrm{colim}}_{f^*S}\, U$, i.e. f^*S is a colim sieve. Therefore, every covering sieve in the canonical topology is a universal colim sieve.

To see that $\mathcal{U}$ is subcanonical, let M be any object in $\mathcal{C}$ and consider the representable presheaf rM. For any $T \in \mathcal{U}(X)$,

$$\begin{aligned}
rM(X) &\cong rM\left(\underrightarrow{\mathrm{colim}}_{T}\, U\right) \\
&\cong \varprojlim_{T}(rM \circ U) \\
&\cong \mathrm{Eq}\left(\prod_{A \xrightarrow{f} X \in T} F(A) \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{\substack{B \xrightarrow{g} A \\ A \xrightarrow{f} X \in T}} F(B)\right)
\end{aligned}$$

where the first isomorphism is because T is a colim sieve, the second isomorphism is a general property of $\mathrm{Hom}_{\mathcal{C}}(-, M)$, and third isomorphim is fact (3.4). Since this is true for every universal colim sieve T and object X, then rM is a sheaf. Therefore, all representable presheaves are sheaves in the universal colim sieve topology. $\square$

3.2.1 Basis

Now, for a very specific type of category, we give a basis for the canonical topology.

Proposition 3.2.2. Let $\mathcal{C}$ be a cocomplete category with pullbacks. Futher assume that coproducts and pullbacks commute in $\mathcal{C}$. Then $S = \langle \{f_\alpha \colon A_\alpha \to X\}_{\alpha \in \mathcal{A}} \rangle$ is a (universal) colim sieve if and only if $T = \langle \{\coprod f_\alpha \colon \coprod_{\alpha \in \mathcal{A}} A_\alpha \to X\} \rangle$ is a (universal) colim sieve.

Proof. Fix $f \colon Y \to X$ and consider f^*S and f^*T. Then

$$\begin{aligned}
\underset{\overrightarrow{f^*T}}{\mathrm{colim}}\, U &\cong \mathrm{Coeq}\begin{pmatrix} \left(\left(\coprod_{\gamma \in \mathcal{A}} A_\gamma\right) \times_X Y\right) \times_Y \left(\left(\coprod_{\beta \in \mathcal{A}} A_\beta\right) \times_X Y\right) \\ \downarrow\downarrow \\ \left(\coprod_{\alpha \in \mathcal{A}} A_\alpha\right) \times_X Y \end{pmatrix} \\
&\cong \mathrm{Coeq}\begin{pmatrix} \left(\coprod_{\gamma \in \mathcal{A}} (A_\gamma \times_X Y)\right) \times_Y \left(\coprod_{\beta \in \mathcal{A}} (A_\beta \times_X Y)\right) \\ \downarrow\downarrow \\ \coprod_{\alpha \in \mathcal{A}} (A_\alpha \times_X Y) \end{pmatrix} \\
&\cong \mathrm{Coeq}\begin{pmatrix} \coprod_{\gamma, \beta \in \mathcal{A}} ((A_\gamma \times_X Y) \times_Y (A_\beta \times_X Y)) \\ \downarrow\downarrow \\ \coprod_{\alpha \in \mathcal{A}} (A_\alpha \times_X Y) \end{pmatrix} \\
&\cong \underset{\overrightarrow{f^*S}}{\mathrm{colim}}\, U
\end{aligned}$$

by Lemma 2.1.1, Proposition 2.1.2 and the commutativity of coproducts and pullbacks. Therefore, $\varinjlim_{f^*S} U \cong Y$ if and only if $\varinjlim_{f^*T} U \cong Y$. □

Theorem 3.2.3. Let $\mathcal{C}$ be a cocomplete category with pullbacks whose coproducts and pullbacks commute. A sieve S on X is a (universal) colim sieve of $\mathcal{C}$ if and only if there exists some $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}} \subset S$ where $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a (universal) effective epimorphism.

Proof. It is an easy application of Proposition 3.2.2, Corollary 2.1.5 and Theorem 3.1.1. □

The above theorem shows us what our basis for the canonical topology should be; and indeed:

Theorem 3.2.4. Let $\mathcal{C}$ be a cocomplete category with stable and disjoint coproducts and all pullbacks. For each X in $\mathcal{C}$, define $K(X)$ by

$$\{A_\alpha \to X\}_{\alpha\in\mathcal{A}} \in K(X) \iff \coprod_{\alpha\in\mathcal{A}} A_\alpha \to X \text{ is a universal effective epimorphism.}$$

Then K is a Grothendieck basis and generates the canonical topology on $\mathcal{C}$.

Proof. We will use the universal colim sieve presentation (Theorem 3.2.1). For K to be a basis we need three things:

1. $\{f\colon E \to X\} \in K(X)$ for every isomorphism f.
2. If $\{f_i\colon E_i \to X\}_{i\in I} \in K(X)$ and $g\colon Y \to X$, then $\{\pi_2\colon E_i \times_X Y \to Y\}_{i\in I}$ is in $K(Y)$
3. If $\{f_i\colon E_i \to X\}_{i\in I} \in K(X)$ and if for each $i \in I$ we have $\{g_{ij}\colon D_{ij} \to E_i\}_{j\in J_i}$ in $K(E_i)$, then $\{f_i \circ g_{ij}\colon D_{ij} \to X\}_{i\in I, j\in J_i} \in K(X)$.

The first condition is true since isomorphisms are obviously universal effective epimorphisms. The second condition follows from the fact that coproducts and pullbacks commute, and the assumed universal condition on $\coprod_{i\in I} E_i \to X$. The third condition follows from Corollary 2.1.7 and Lemma 2.1.10.

Lastly, Theorem 3.2.3 showcases that this Grothendieck basis is indeed a basis for the canonical topology. $\square$

CHAPTER IV

GIRAUD'S THEOREM AND THE CANONICAL TOPOLOGY

Giraud's Theorem shows that categories with certain nice properties can be written as sheaves on a Grothendieck site. We show that in fact, modulo universe considerations, one may take this site to be the original category with the canonical topology.

We will specifically use the version of Giraud's Theorem stated in [10]. In fact, the appendix of [10] has a thorough discussion of Giraud's theorem and all of the terminology used in it; we will include the basics of this discussion for completeness. We will begin by recalling the definitions used in Maclane's and Moerdijk's version of Giraud's Theorem.

Throughout this section, let $\mathcal{E}$ be a category with small hom-sets and all finite limits.

Coequalizer Morphisms and Kernel Pairs

Definition 4.0.1. We call a morphism $f\colon Y \to Z$ in $\mathcal{E}$ a *coequalizer* if there exists some object X and morphisms $\partial_0, \partial_1 \colon X \to Y$ such that

$$X \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} Y \xrightarrow{f} Z$$

is a coequalizer diagram.

We remark that every coequalizing morphism is an epimorphism but the converse of this statement is not guaranteed.

Definition 4.0.2. The pair of morphisms $\partial_0, \partial_1 \colon X \to Y$ are called a *kernel pair* for $f \colon Y \to Z$ if the following is a pullback diagram

$$\begin{tikzcd} X \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

Equivalence Relations and Quotients

Definition 4.0.3. An *equivalence relation* on the object E of $\mathcal{E}$ is a subobject R of $E \times E$, represented by the monomorphism $(\partial_0, \partial_1) \colon R \to E \times E$, satisfying the following axioms

1. (reflexive) the diagonal $\Delta \colon E \to E \times E$ factors through (∂_0, ∂_1),

2. (symmetric) the map $(\partial_1, \partial_0) \colon R \to E \times E$ factors through (∂_0, ∂_1),

3. (transitivity) if $R \times_E R$ is the pullback

$$\begin{tikzcd} R \times_E R \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] & R \arrow[d, "\partial_0"] \\ R \arrow[r, "\partial_1"'] & E \end{tikzcd}$$

then $(\partial_1 \pi_1, \partial_0 \pi_0) \colon R \times_E R \to E \times E$ factors through R.

Definition 4.0.4. If E is an object of $\mathcal{E}$ with equivalence relation R, then the *quotient* is denoted E/R and is defined to be

$$\mathrm{Coeq}\left(R \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} E \right)$$

provided that this coequalizer exists.

Stably Exact Forks

A diagram is called a *fork* if it is of the form

$$X \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} Y \xrightarrow{q} Z. \tag{4.1}$$

Definition 4.0.5. The fork (4.1) is called *exact* if ∂_0 and ∂_1 are the kernel pair for q, and q is the coequalizer of ∂_0 and ∂_1.

Definition 4.0.6. The fork (4.1) is called *stably exact* if the pullback of (4.1) along any morphism in $\mathcal{E}$ yields an exact fork, i.e. if for any $Z' \to Z$ in $\mathcal{E}$,

$$X \times_Z Z' \rightrightarrows Y \times_Z Z' \xrightarrow{q \times 1} Z \times_Z Z'$$

is an exact fork.

Generating Sets

Definition 4.0.7. A set of objects $\{A_i \,|\, i \in I\}$ of $\mathcal{E}$ is said to *generate* $\mathcal{E}$ if for every object E of $\mathcal{E}$, $W = \{A_i \to E \,|\, i \in I\}$ is an epimorphic family (in the sense that for any two parallel arrows $u, v \colon E \to E'$, if every $w \in W$ yields the identity $uw = vw$, then $u = v$).

Giraud's Theorem

Theorem 4.0.8 (Giraud, [10])**.** A category $\mathcal{E}$ with small hom-sets and all finite limits is a Grothendieck topos if and only if it has the following properties (which we will refer to as *Giraud's axioms*):

(i) $\mathcal{E}$ has small coproducts which are disjoint and stable under pullback,

(ii) every epimorphism in $\mathcal{E}$ is a coequalizer,

(iii) every equivalence relation $R \rightrightarrows E$ in $\mathcal{E}$ is a kernel pair and has a quotient,

(iv) every exact fork $R \rightrightarrows E \to Q$ is stably exact,

(v) there is a small set of objects of $\mathcal{E}$ which generate $\mathcal{E}$.

Discussion 4.0.9. Taken together, Giraud's axioms (ii) and (iv) imply that for each epimorphism $B \xrightarrow{f} A$, the fork $B \times_A B \rightrightarrows B \to A$ is stably exact. The exactness implies f is an effective epimorphism and the stability implies f is a universal effective epimorphism.

Notation 4.0.10. We use $Sh(\mathcal{E}, J)$ to represent the category of sheaves on the category $\mathcal{E}$ under the topology J.

Suppose the category $\mathcal{E}$ has small hom-sets and all finite limits, satisfies Giraud's axioms, and whose small set of generators (axiom v) is $\mathcal{C}$. In [10] Mac Lane and Moerdijk specifically prove $\mathcal{E} \cong Sh(\mathcal{C}, J)$ where J is the Grothendieck topology on $\mathcal{C}$ defined by:

$$S \in J(X) \text{ if and only if } \coprod_{(g\colon D \to X) \in S} D \to X \text{ is an epimorphism in } \mathcal{E}.$$

(In particular, Mac Lane and Moerdijk prove that J is a Grothendieck topology.)

Corollary 4.0.11. Suppose the category $\mathcal{E}$ has small hom-sets and all finite limits, satisfies Giraud's axioms, and whose small set of generators (axiom v) is $\mathcal{C}$. Then $\mathcal{E}$ is equivalent to $Sh(\mathcal{C}, C)$ where C is the canonical topology on $\mathcal{C}$.

Proof. Let J be the topology defined above. Additionally, the above discussion implies that it suffices to show that J is the canonical topology. By Theorem 3.2.1, we will instead show that every universal colim sieve is in J and that every sieve in J is a universal colim sieve.

By Remark 2.1.8, coproducts and pullbacks commute and hence for any collection of morphisms $\{A_i \to X\}_{i \in I}$ in $\mathcal{E}$, the diagrams

$$\begin{array}{c} \coprod_{I^2}(A_i \times_X A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array} \quad \text{and} \quad \begin{array}{c} (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}$$

are isomorphic. Note: in both diagrams, the two maps down are the obvious ones induced/obtained from a pullback diagram. Thus

$$\mathrm{Coeq}\left(\begin{array}{c} \coprod_{I^2}(A_i \times_X A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}\right) \cong \mathrm{Coeq}\left(\begin{array}{c} (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}\right).$$

But by Proposition 2.1.2 (which is usable since $\mathcal{E}$ is cocomplete),

$$\mathrm{Coeq}\left(\begin{array}{c} \coprod_{I^2}(A_i \times_X A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}\right) \cong \varinjlim_{S} U \quad \text{where } S = \langle \{A_i \to X\}_{i \in I} \rangle$$

and

$$\mathrm{Coeq}\left(\begin{array}{c} (\coprod_I A_i) \times_X (\coprod_I A_j) \\ \downarrow\downarrow \\ \coprod_I A_k \end{array}\right) \cong \varinjlim_{T_S} U \quad \text{where } T_S = \left\langle \left\{ \left(\coprod_I A_i \right) \to X \right\} \right\rangle.$$

Hence

$$\varinjlim_{S} U \cong \varinjlim_{T_S} U$$

$$\text{where } S = \langle \{A_i \to X\}_{i \in I} \rangle \quad \text{and} \quad T_S = \left\langle \left\{ \left(\coprod_I A_i \right) \to X \right\} \right\rangle \tag{4.2}$$

for any generating set $\{A_i \to X\}_{i \in I}$ of S.

Suppose S is a universal colim sieve. Since S has the some generating set, then by the definition of colim sieve and (4.2),

$$X \cong \varinjlim_{S} U \cong \varinjlim_{T_S} U.$$

This implies that T_S is a colim sieve. Hence $\left(\coprod_{(g\colon D \to X) \in S} D \right) \to X$ is an effective epimorphism by Corollary 2.1.5 and so $S \in J(X)$.

For the converse, suppose that $S \in J(X)$. Thus $p_s\colon \left(\coprod_{(g\colon D \to X) \in S} D \right) \to X$ is an epimorphism, which by Discussion 4.0.9 is a universal effective epimorphism. Hence by Corollary 2.1.5, p_s generates a universal colim sieve called T_S. Then by the definition of colim sieve and (4.2),

$$X \cong \varinjlim_{T_S} U \cong \varinjlim_{S} U.$$

Therefore S is a colim sieve.

Similar to the last paragraph, we can use (4.2) to show that f^*S is a colim sieve for any morphism f in $\mathcal{E}$ if we know that T_{f^*S} is a colim sieve. So to finish the proof we will use the fact that T_S is a universal colim sieve to show that T_{f^*S} is a colim sieve. Let $f\colon Y \to X$ be any morphism in $\mathcal{E}$. Then by using S as a generating collection for itself and Lemma 2.1.1, $f^*S = \langle\{A \times_X Y \to Y \mid A \to X \in S\}\rangle$. Similarly, using Lemma 2.1.1, $f^*T_S = \left\langle\left\{\left(\coprod_{(A\to X\in S)} A\right) \times_X Y \to Y\right\}\right\rangle$. Then by Remark 2.1.8

$$\coprod_{(A\to X)\in S} (A \times_X Y) \cong \left(\coprod_{(A\to X)\in S} A\right) \times_X Y$$

over Y. Therefore,

$$\varinjlim_{T_{f^*S}} U \cong \varinjlim_{f^*T_S} U \cong Y$$

where the first isomorphism is due to the previous few sentences and the second isomorphism is due to the fact that T_S is a universal colim sieve. Thus T_{f^*S} is a colim sieve. $\square$

CHAPTER V

UNIVERSAL COLIM SIEVES IN THE CATEGORIES OF SETS AND TOPOLOGICAL SPACES

In this section we examine the canonical topology on the categories of sets, all topological spaces and compactly generated weakly Haudsdorff spaces.

Notation 5.0.1. We will use **Sets** to denote the category of sets. We will use **Top** to denote the category of all topological spaces, **CG** to denote the category of compactly generated spaces, and **CGWH** to denote the category of compactly generated weakly Hausdorff spaces. When we want to talk about the category of topological spaces without differentiating between **Top** and **CGWH**, then we will use **Spaces**; all results about **Spaces** will hold for both **Top** and **CGWH**.

We will begin with a few reminders about the category of compactly generated weakly Hausdorff spaces based on the references [15] and [11]. Specifically, there are functors $k\colon \mathbf{Top} \to \mathbf{CG}$ and $h\colon \mathbf{CG} \to \mathbf{CGWH}$ such that

- For a topological space X with topology τ, a subset Y of X is called k-closed if $u^{-1}(Y)$ is closed in K for every continuous map $u\colon K \to X$ and compact Hausdorff space K. The collection of all k-closed subsets, called $k(\tau)$, is a topology.
- The functor k takes X with topology τ to the set X with topology $k(\tau)$.
- k is right adjoint to the inclusion functor $\iota\colon \mathbf{CG} \to \mathbf{Top}$.
- $h(X)$ is X/E where E is the smallest equivalence relation on X closed in $X \times X$.
- h is left adjoint to the inclusion functor $\iota'\colon \mathbf{CGWH} \to \mathbf{CG}$.

- A limit in **CGWH** is k applied to the limit taken in **Top**, i.e. for a diagram $F\colon I \to \mathbf{CGWH}$, the limit of F is $k(\lim_I \iota\iota' F)$.

- A colimit in **CGWH** is h applied to the colimit taken in **Top**, i.e. for a diagram $F\colon I \to \mathbf{CGWH}$, the colimit of F is $h(\varinjlim_I \iota\iota' F)$.

Proposition 5.0.2. Let S be a sieve on X in either **Sets** or **Top**. Let $C = \varinjlim_{S} U$. Then the natural map $\varphi\colon C \to X$ is an injection.

Proof. Suppose $\tilde{y}, \tilde{z} \in C$ and $\varphi(\tilde{y}) = x = \varphi(\tilde{z})$. We can pick a $(Y \to X) \in S$ and a $y \in Y$ that represents $\tilde{y}$, i.e. where $y \mapsto \tilde{y}$ under the natural map $Y \to C$; similarly, we can pick a $(Z \to X) \in S$ and a $z \in Z$ representing $\tilde{z}$. Then the inclusion $i\colon \{x\} \hookrightarrow X$ factors through both Y and Z by $x \mapsto y$ and $x \mapsto z$ respectively. Thus $i \in S$. Hence $\tilde{y} = \tilde{z}$ in C. $\square$

Corollary 5.0.3. Let S be a sieve on X in **CGWH**. Then the colimit over S taken in **Top** is in **CGWH**, i.e. $h(\varinjlim_I \iota\iota' U) = \varinjlim_I \iota\iota' U$. Moreover, the natural map $\varphi\colon \varinjlim_S U \to X$ is an injection.

Proof. We will make use of the following Proposition from [15]: if Z is in **CG**, then Z is weakly Hausdorff if and only if the diagonal subspace Δ_Z is closed in $Z \times Z$. Additionally, we remark that colimits of compactly generated spaces computed in **Top** are automatically compactly generated.

Let $C = \varinjlim_S \iota\iota' U$, i.e. C is the colimit over S taken in **Top**. By Proposition 5.0.2, the natural map $\varphi\colon C \to X$ is an injection; we remark that it is not the statement of Proposition 5.0.2 that gives this observation since S is not a sieve in Top, instead the proof of Proposition 5.0.2 holds in this situation since $\{x\}$ is in **CGWH**. Since X is **CGWH**, then Δ_X is closed in $X \times X$. Since φ is a continuous injection, then $(\varphi \times \varphi)^{-1}(\Delta_X) = \Delta_C$ is closed in $C \times C$. $\square$

5.1 Basis and Presentation

Recall Theorem 3.2.4: Let $\mathcal{C}$ be a cocomplete category with stable and disjoint coproducts and all pullbacks. Then $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ is part of the basis for the canonical topology if and only if $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a universal effective epimorphism.

Additionally recall Theorem 3.2.3: Let $\mathcal{C}$ be a cocomplete category with pullbacks whose coproducts and pullbacks commute. A sieve S on X is a (universal) colim sieve of $\mathcal{C}$ if and only if there exists some $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}} \subset S$ where $\coprod\limits_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a (universal) effective epimorphism.

The categories **Sets**, **Top** and **CGWH** all satisfy the hypotheses of Theorems 3.2.4 and 3.2.3. Thus we have the following corollaries of Theorems 3.2.4 and 3.2.3 based on what the universal effective epimorphisms are in each category.

Proposition 5.1.1. In **Sets**, $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ is part of a basis for the canonical topology if and only if $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a surjection. In particular, a sieve of the form $S = \langle\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}\rangle$ on X is in the canonical topology if and only if $\coprod\limits_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a surjection. Moreover, every colim sieve is universal.

Proof. It is easy to see in **Sets** that the effective epimorphisms are precisely the surjections. Since pulling back a surjection yields a surjection, then the universal effective epimorphisms in the category of sets are also the surjections. Lastly, this implies, by Theorem 3.2.3, that every colim sieve is universal. □

Remark 5.1.2. Since **Sets** is a Grothendieck topos, we can compare Proposition 5.1.1 to the proof of Corollary 4.0.11. Specifically, Proposition 5.1.1 allows us to determine if a sieve is in the canonical topology by looking only at the sieve's generating set whereas the proof of Corollary 4.0.11 along with the Grothendieck topology J require us to look at the entire sieve.

Recall that a quotient map f is called *universal* if every pullback of f along a map yields a quotient map.

Proposition 5.1.3. In **Top**, $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ is part of a basis for the canonical topology if and only if $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a universal quotient map. Additionally, a sieve S on X is a (universal) colim sieve if and only if there exists some collection $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}} \subset S$ such that $\displaystyle\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a (universal) quotient map. In particular, $T = \langle\{f\colon Y \to X\}\rangle$ is a (universal) colim sieve if and only if f is a (universal) quotient map.

Proof. It is a well-known fact that in **Top** the effective epimorphisms are precisely the quotient maps. □

Proposition 5.1.4. In **CGWH**, $\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}$ is part of the basis for the canonical topology if and only if $\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a quotient map. In particular, a sieve $S = \langle\{A_\alpha \to X\}_{\alpha\in\mathcal{A}}\rangle$ on X is in the canonical topology if and only if $\displaystyle\coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ is a quotient map. Moreover, every colim sieve is universal.

Proof. This is a consequence of Corollary 2.1.5, Corollary 5.0.3, the fact that the universal effective epimorphisms in **Top** are precisely the universal quotient maps, and [15, Proposition 2.36], which states that every quotient map in **CGWH** is universal.

□

5.2 Examples in the category of Spaces

In this section we will use our basis to talk about some specific examples; including a special circumstance (when a sieve is generated by one function) and how the canonical topology on the categories **CGWH** and **Top** can differ in this situation.

Definition 5.2.1. For a category D, we call $\mathfrak{A} \subset \text{ob(D)}$ a *weakly terminal set* of D if for every object X in D, there exists some $A \in \mathfrak{A}$ and morphism $X \to A$ in D.

Additionally, if $F\colon D \to C$ is a functor and D has a weakly terminal set $\mathfrak{A}$, then we call $\{F(A)\}_{A\in\mathfrak{A}}$ a *weakly terminal set* of F.

For example, if $S = \langle\{A_\alpha \to X\}_{\alpha\in\mathfrak{A}}\rangle$ is a sieve on X then $\{A_\alpha\}_{\alpha\in\mathfrak{A}}$ is the weakly terminal set of U. Or as another example, $\{Y\}$ is the weakly terminal set of the diagram $Y \times_X Y \rightrightarrows Y$. One easy consequence of this in **Top** is a reduction of the colimit topology: V is open in the colimit if and only if the preimage of V is open in each member of the weakly terminal set.

Proposition 5.2.2. Let $F\colon D \to$ **Spaces** be a functor where D has a weakly terminal set $\mathfrak{A}$. Suppose $f_A\colon F(A) \to X$ is an open map for all $A \in \mathfrak{A}$, then the induced map $\varphi\colon \varinjlim_D F \to X$ is an open map. Similarly, if the f_A are all closed and $\mathfrak{A}$ is a finite set, then φ is a closed map.

Proof. Let $C = \varinjlim F$ and $i_A\colon F(A) \to C$ be the natural maps. Both results follow from the easy set equality below for $B \subset C$

$$\varphi(B) = \bigcup_{A\in\mathfrak{A}} f_A(i_A^{-1}(B))$$

since i_A^{-1}, f_A and unions respect open/closed sets in their respective scenarios. □

Corollary 5.2.3. Let $S = \langle\{f_\alpha\colon A_\alpha \to X\}_{\alpha\in\mathcal{A}}\rangle$ be a sieve on X in **Spaces** with the induced map $\eta\colon \coprod_{\alpha\in\mathcal{A}} A_\alpha \to X$ a surjection. If all of the f_α are open maps or if $\mathcal{A}$ is a finite collection and all of the f_α are closed maps, then S is a colim sieve.

Proof. Let $\varphi\colon \varinjlim_S U \to X$ be the natural map. By Proposition 5.0.2, Corollary 5.0.3, and the surjectivity of η, φ is a continuous bijection. Then Proposition 5.2.2 implies that φ is open or closed, depending on the case, and hence an isomorphism. □

This corollary leads us to some nice examples of sieves we would hope are in the canonical topology and actually are!

Example 5.2.4. Let X be any space and let $\{U_i\}_{i\in I}$ be an open cover of X. Then the inclusion maps $U_i \hookrightarrow X$ generate a universal colim sieve, call it S. Indeed, by Corollary 5.2.3, S is a colim sieve. Universality is obvious, as the preimage of an open cover is an open cover.

Example 5.2.5. Let X be any space and let $K_1, \ldots, K_n$ be a closed cover of X. For the exact same reasons as the previous example, the inclusions $K_i \hookrightarrow X$ generate a sieve in the canonical topology.

Before we give our next example, we rephrase [2, Theorem 1], which completely characterizes universal quotient maps in **Top**:

Theorem 5.2.6 (Day and Kelly, 1970)**.** Let $f\colon Y \to X$ be a quotient map. Then f is a universal quotient map if and only if for every $x \in X$ and cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of $f^{-1}(x)$ by opens in Y, there is a finite set $\{\alpha_1, \ldots, \alpha_n\} \subset \Lambda$ such that $fG_{\alpha_1} \cup \cdots \cup fG_{\alpha_n}$ is a neighborhood of x.

Example 5.2.7. Consider the diagram $B_1 \to B_2 \to B_3 \to \ldots$ and the direct limit $B = \varinjlim B_n$ in **Top**. Let $S = \langle\{\iota_n\colon B_n \to B \,|\, n \in \mathbb{N}\}\rangle$ where ι_n are the natural maps into the colimit. By Proposition 5.1.3, S is a colim sieve because $\coprod_{n\in\mathbb{N}} B_n \to B$ is obviously a quotient map. However, S is not necessarily in the canonical topology – we can use Proposition 5.1.3 on specific examples to see when S is and is not in the canonical topology.

For example, suppose there exists an N such that $B_m = B_N$ whenever $m > N$. Then $B = B_N$. Hence it is easy to see by Day and Kelly's condition that the map

$\coprod_{n\in\mathbb{N}} B_n \to B$ is a universal quotient map. Therefore, the S from this example is in the canonical topology.

As another example, take $B_n = \mathbb{R}^n$ and let $B_n \to B_{n+1}$ be the closed inclusion map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. Use $\mathbb{R}^\infty$ to denote the direct limit. We claim that $\coprod_{n\in\mathbb{N}} \mathbb{R}^n \to \mathbb{R}^\infty$ is not a universal quotient map. Indeed, consider Day and Kelly's condition; take $x = 0 \in \mathbb{R}^\infty$ and the open cover in $\coprod_{n\in\mathbb{N}} \mathbb{R}^n$ consisting of open disks $D^n \subset \mathbb{R}^n$ centered at the origin with fixed radius $\epsilon > 0$. Pick any finite collection $D^{n_1}, \ldots, D^{n_k}$ with $n_1 < \cdots < n_k$. Then for $i = 1, \ldots, k$ we can view D^{n_i} as a subset of $\mathbb{R}^{n_k}$. Hence $\cup_{i=1}^k \iota_{n_i}(D^{n_i}) = \cup_{i=1}^k \iota_{n_k}(D^{n_i}) \subset \iota_{n_k}(\mathbb{R}^{n_k})$. However, by dimensional considerations, we can see that for all $b \in \mathbb{N}$, $\iota_b(\mathbb{R}^b)$ contains no open sets of $\mathbb{R}^\infty$ and hence $\cup_{i=1}^k \iota_{n_i}(D^{n_i})$ cannot be a neighborhood of x in $\mathbb{R}^\infty$. Remark: To see that $\iota_b(\mathbb{R}^b)$ contains no open sets, suppose to the contrary and call the open set V. Then $\iota_{b+1}^{-1}(V)$ is open in $\mathbb{R}^{b+1}$ and in particular, contains an open ball of dimension $b+1$. Thus dimensional considerations imply that $\iota_{b+1}^{-1}(V)$ is not contained in the image of $\mathbb{R}^b$ in $\mathbb{R}^{b+1}$. Since each ι_n is an inclusion map, then $\iota_{b+1}\iota_{b+1}^{-1}(V) \not\subset \iota_{b+1}(\mathbb{R}^b)$ and so V is not contained in $\iota_b(\mathbb{R}^b)$, which is our contradiction. Therefore, the S from this example is not in the canonical topology.

Example 5.2.8. Consider the diagram $B_1 \to B_2 \to B_3 \to \ldots$ and the direct limit $B = \varinjlim B_n$ in **CGWH**. Let $S = \langle \{\iota_n \colon B_n \to B \mid n \in \mathbb{N}\} \rangle$ where ι_n are the natural maps into the colimit. Then by Proposition 5.1.4, S is a universal colim sieve because $\coprod_{n\in\mathbb{N}} B_n \to B$ is a quotient map.

Now we shift our focus to sieves that can be generated by one map, called *monogenic sieves*. There are many reasons one could focus on these kinds of sieves, however by Proposition 3.2.2, if we fully comprehend when monogenic sieves are in the canonical topology, then we can (in some sense) completely understand the canonical topology. From this point onward, this section will be about monogenic sieves; in other words, by Proposition 5.1.3 and Proposition 5.1.4, we will be focusing on (universal) quotient maps.

Remark 5.2.9. Some examples will talk about the space $\mathbb{R}/\mathbb{Z}$. In this section, this space is not a group quotient but instead is the squashing of the subspace $\mathbb{Z}$ to a point.

Example 5.2.10. Consider the quotient maps $f\colon S^n \to \mathbb{R}P^n$ and $g\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. There is some subtly, which will depend on the category we are in, in determining if f or g generate universal colim sieves. Throughout the rest of this section we will continue to explore this particular example.

Monogenic Sieves in **CGWH**

By Proposition 5.1.4, if X and Y are in **CGWH** and $h\colon Y \to X$, then $\langle \{h\} \rangle$ is in the canonical topology if and only if h is a quotient map. Therefore, we immediately get the following examples:

Example 5.2.11. Topological manifolds are in **CGWH**. Thus S^n and $\mathbb{R}P^n$ are in **CGWH**. Hence $\langle \{f\colon S^n \to \mathbb{R}P^n\} \rangle$ is in the canonical topology.

Example 5.2.12. Every CW-complex is in **CGWH**. Thus $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are in **CGWH**. Hence $\langle \{g\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}\} \rangle$ is in the canonical topology.

Monogenic Sieves in **Top**

This section will heavily rely on Theorem 5.2.6 (the Theorem by Day and Kelly characterizing universal quotient maps in **Top**) because a monogenic sieve generated by f is in the canonical topology if and only if f is a universal quotient map.

Example 5.2.13. Day and Kelly's theorem implies that every open quotient map is a universal quotient map. Therefore, the quotient map $f\colon S^n \to \mathbb{R}P^n$ is a universal quotient map and $\langle\{f\colon S^n \to \mathbb{R}P^n\}\rangle$ is in the canonical topology.

Example 5.2.14. The quotient map $g\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is not universal. We will demontrate this in two ways, first by using Day and Kelly's theorem and second by directly showing g is not universal. Note: many sets of $\mathbb{R}/\mathbb{Z}$ will be written as if they are in $\mathbb{R}$ for ease of presentation.

(i) We will look at Day and Kelly's condition for $\mathbb{Z} \in \mathbb{R}/\mathbb{Z}$ with the open cover (in $\mathbb{R}$) $\{G_i := (i-m, i+m)\}_{i\in\mathbb{Z}}$ for a fixed $m \in \left(0, \frac{1}{2}\right)$. For any open set U of $\mathbb{R}/\mathbb{Z}$ containing $\mathbb{Z}$, the quotient topology tells us that $g^{-1}(U)$ is an open neighborhood of $\mathbb{Z} \subset \mathbb{R}$. But for any n, $g^{-1}(\bigcup_{k=1}^{n} gG_{i_k}) = \mathbb{Z} \cup (\bigcup_{k=1}^{n}(i_k - m, i_k + m))$ is not a neighborhood of $\mathbb{Z} \subset \mathbb{R}$. So there cannot be any open set of $\mathbb{R}/\mathbb{Z}$ containing $\mathbb{Z}$ that is contained in $\bigcup_{k=1}^{n} gG_{i_k}$ for any finite collection of the cover.

(ii) To directly show that g is not universal we need to come up with a space and map to $\mathbb{R}/\mathbb{Z}$ where g pulledbacked along this map is not a quotient map. Our candidate is the following: Let $t(\mathbb{R}/\mathbb{Z})$ be the set $\mathbb{R}/\mathbb{Z}$ with the topology where U (written as if it is in $\mathbb{R}$) is said to be open if (a) $\mathbb{Z} \not\subset U$ or (b) U contains $\mathbb{Z}$ and is a neighborhood (in the typical topology) of ($\mathbb{Z}-\{$finitely many or no points$\}$). Remark: this topology was used in Day and Kelly's paper (in the proof of their theorem), however they defined the topology using a filter and we have merely rephrased it for convenience.

Define $\kappa\colon t(\mathbb{R}/\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ by the set identity map; this is a continuous map. As a set, the pullback of domain(g) along κ is $\mathbb{R}$ but since it now has the limit topology, we denote the pullback as $t(\mathbb{R})$; in particular, $t(\mathbb{R})$ is $\mathbb{R}$ with the discrete topology. Denote the projection maps as $g'\colon t(\mathbb{R}) \to t(\mathbb{R}/\mathbb{Z})$ and $\kappa'\colon t(\mathbb{R}) \to \mathbb{R}$.

We claim that g' is not a quotient map, i.e. there is some non-open set B in $t(\mathbb{R}/\mathbb{Z})$ with $(g')^{-1}(B)$ open in $t(\mathbb{R})$. Since every $(g')^{-1}(B)$ is open in $t(\mathbb{R})$, then we merely need to find a B that is not open in $t(\mathbb{R}/\mathbb{Z})$; $B = \{\mathbb{Z}\}$ obviously works.

Here we have our first example of a colim sieve that is not universal – it is even an example using Hausdorff spaces. Additionally, this example shows us that quotient maps of the form $X \to X/A$ may not generate universal colim sieves. So let's understand these special quotient maps a little better. Specifically, using Day and Kelly's theorem, we can completely state what kinds of subspaces A yield universal quotient maps $X \to X/A$:

Corollary 5.2.15. The quotient map $\pi\colon X \to X/A$ is universal if and only if both of the following properties hold:

1. If A is not open, then for every open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of $(\partial A)\cap A$ in X there is a finite collection $\{\alpha_1, \ldots, \alpha_n\} \subset \Lambda$ with $A \cup G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$ open in X.

2. If A is not closed, then for every open U in X such that $U \cap (\overline{A} - A) \neq \emptyset$, $U \cup A$ is open in X.

Proof. We will be using Theorem 5.2.6 in two ways: first by finding the necessary conditions for π to be a universal quotient map (i.e. proving the forward direction) and then second by checking the sufficient conditions in the three cases (i) $x = A$, (ii) $x \in X - \overline{A}$, and (iii) $x \in \overline{A} - A$ (i.e. proving the backward direction).

First suppose that π is a universal quotient map. To see that the first property is necessary, assume that $(\partial A) \cap A \neq \emptyset$, i.e. A is not open, and we have an open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of $(\partial A)\cap A$. Then we can expand this cover to an open cover of A by adding $Int(A)$ to $\{G_\alpha\}_{\alpha\in\Lambda}$. Now by assumption (using the point A in X/A) there is a finite subcollection $G_{\alpha_1}, \ldots, G_{\alpha_n}, Int(A)$ such that $\pi G_{\alpha_1} \cup \cdots \cup \pi G_{\alpha_n} \cup \pi Int(A)$ is a neighborhood of A in X/A. But $\pi Int(A) \subset \pi G_\alpha$ since $G_\alpha \cap A \neq \emptyset$ and so $Int(A)$ is not necessary in our finite subcollection. Thus $\pi G_{\alpha_1} \cup \cdots \cup \pi G_{\alpha_n}$ is a neighborhood of A; let U be an open subset of $\pi G_{\alpha_1} \cup \cdots \cup \pi G_{\alpha_n}$ containing A. Now by looking at the preimages of U and $\bigcup_{i=1}^n \pi G_{\alpha_i}$ in X, we get that

$$A \subset \pi^{-1}(U) \subset \pi^{-1}(\bigcup_{i=1}^{n} \pi G_{\alpha_i}) = G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \cup A.$$

Since $\pi^{-1}(U)$ is open, then the above expression implies $A \subset Int(G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \cup A)$. But since all of the G_α are open, then $G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} \cup A$ is open. Therefore, the first property is necessary.

To see that the second property is necessary, assume that A is not closed and U is any open neighborhood of a fixed $x \in \overline{A} - A$ in X. Since U is an open cover of $\pi^{-1}(\pi(x)) = x$, then by Theorem 5.2.6, πU is a neighborhood of x; let V be an open subset of πU that contains x. Then by looking at the preimages of V and πU, we see (using that U intersects A nontrivially) that

$$A \subset \pi^{-1}(V) \subset \pi^{-1}(\pi U) = U \cup A.$$

But since $\pi^{-1}(V)$ is open, then $A \subset Int(U \cup A)$, i.e. $U \cup A$ is open. Therefore, the second condition is necessary.

Second let's assume the two conditions hold. We will show π is a universal quotient map by checking that the conditions of Theorem 5.2.6 hold in all three locations in X/A (i.e. for (i) $x = A$, (ii) $x \in X - \overline{A}$, and (iii) $x \in \overline{A} - A$).

(i) For $A \in X/A$, take any open cover $\{G_\alpha\}_{\alpha \in \Lambda}$ of A in X. If A is open in X, then $\{A\}$ is open in X/A and hence every πG_α is a neighborhood. If A is not open, let Γ be the finite portion of Λ that property 1 guarantees exists, i.e. $A \cup \left(\bigcup_{i \in \Gamma} G_{\alpha_i}\right)$

is open in X and each G_{α_i} intersects A nontrivially. This implies that $\bigcup_{i\in\Gamma} \pi G_{\alpha_i}$ is an open neighborhood of A in X/A (since its preimage is $A \cup \left(\bigcup_{i\in\Gamma} G_{\alpha_i}\right)$).

(ii) Any $x \in X - \overline{A}$ has an open neighborhood $U_x \subset X - \overline{A}$. Notice that π is a homeomorphism on $X - \overline{A}$. Thus for any such x and any open cover W of $\pi^{-1}(x) = x$ in X, πW is a neighborhood of x because the open neighborhood (in X/A) $U_x \cap W$ is contained in πW.

(iii) If A is closed, then this is trivial so assume that A is not closed and let $x \in \overline{A} - A$. For any open cover W of $\pi^{-1}(x) = x$ in X, $\pi^{-1}(\pi W) = W \cup A$, which is open in X by condition 2. Thus πW is an open neighborhood of x in X/A.

Therefore, our two conditions ensure that π satisfies Day and Kelly's universal quotient map condition. $\square$

Corollary 5.2.15 now gives us a way to produce more examples of sieves in the canonical topology:

Example 5.2.16. Every quotient of a Hausdorff space by a compact subspace is universal. For example, $\pi\colon D^n \to S^n$ (where $S^n = D^n/\partial D^n$) generates a universal colim sieve.

Example 5.2.17. If A is closed, then $S = \langle\{X \to X/A\}\rangle$ is always a colim sieve. Moreover, it is universal if and only if ∂A is compact. For example, this tells us $\langle\{\mathbb{R} \to \mathbb{R}/[0,\infty)\}\rangle$ is in the canonical topology and reaffirms that $\langle\{\mathbb{R} \to \mathbb{R}/\mathbb{Z}\}\rangle$ is not.

CHAPTER VI

UNIVERSAL COLIM SIEVES IN THE CATEGORY OF R-MODULES

The category of R-modules does not satisfy the assumptions of Theorem 3.2.3 or Theorem 3.2.4. Indeed, coproducts and pullbacks of R-modules do not commute (for example, let $\mathbb{Z}_{(a,b)}$ denote the domain of $\mathbb{Z} \to \mathbb{Z}^2$, $1 \mapsto (a,b)$, then we see that $(\mathbb{Z}_{(1,0)} \oplus \mathbb{Z}_{(0,1)}) \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)} \cong \mathbb{Z}$ but $(\mathbb{Z}_{(1,0)} \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)}) \oplus (\mathbb{Z}_{(0,1)} \times_{\mathbb{Z}^2} \mathbb{Z}_{(1,1)}) \cong 0$). Thus we do not have basis and presentation results. Instead, we have some smaller results, reductions and examples.

Notation 6.0.1. Let R be a commutative ring with identity. We will use R-**Mod** for the category of R-modules and **Ab** for the category of abelian groups.

We start with some basic results.

Corollary 6.0.2. Any sieve containing a universal effective epimorphism (e.g. a surjection in R-**Mod** or in **Sets**) is a universal colim sieve.

Proof. This is an immediate consequence of Theorem 3.1.1 and Corollary 2.1.5. $\square$

Lemma 6.0.3. In R-**Mod**, if a sieve S on X can be generated by at most two morphisms, then the canonical map $c\colon \varinjlim_S U \to X$ is an injection.

Proof. Suppose $S = \langle\{f\colon Y \to X, g\colon Z \to X\}\rangle$ and $c(x) = 0$. Since every map in S either factors through f or g, then x, as an element of $\bigoplus_{A\to X\in S} A$, is really an element $(y,z) \in Y \oplus Z$ in the colimit. So $c(x) = 0$ implies that $y + z = 0$ in X, i.e. $(y,-z) \in Y \times_X Z$. Thus $y \in Y$ gets identified with $-z \in Z$ in the colimit; hence $(y,z) = (0, z - z) = 0$ in the colimit. Therefore, $x = 0$ in the colimit and the map c is an injection.

Using the fact that $\langle\{A_i \to X\}_\alpha\rangle = \langle\{A_i \to X\}_\alpha \cup \{Z \xrightarrow{0} X\}\rangle$, we can say that any sieve generated by one morphism is also generated by two morphsims. This completes the proof. $\square$

Proposition 6.0.4. In R-**Mod**, let

$$S = \langle\{f\colon Y \to X\}\rangle \qquad \text{and} \qquad T = \langle\{g\colon U \to X, h\colon V \to X\}\rangle$$

be sieves on X. Then

1. S is a universal colim sieve if and only if f is a surjection.

2. T is a colim sieve if and only if $g \oplus h\colon U \oplus V \to X$ is a surjection.

Proof. For part 2, Lemma 6.0.3 tells us that we only need to worry about the surjectivity of $\varinjlim_T U \to X$ but this is exactly what the above condition is.

For part 1, Lemma 6.0.3 and Lemma 2.1.1 tell us that we only need worry about the surjectivity of $A \times_X Y \xrightarrow{\pi_1} A$ (the generator of k^*S) for every map $k\colon A \to X$. But $A \times_X Y = \{(a, y) \in A \times Y \mid k(a) = f(y)\}$. Hence π_1 is a surjection for every map k if and only if f is a surjection. $\square$

Lemma 6.0.5. In R-**Mod**, suppose $S = \langle\{f_i\colon M_i \to R\}_{i\in I}\rangle$ is a sieve on R such that for every $i \in I$ there exists an $a_i \in R$ with $im(f_i) = a_iR$. If $(a_i \mid i \in I) = R$, then for every R-module homomorphism $g\colon N \to R$, the natural map $\varinjlim_{g^*S} U \to N$ is a surjection.

Proof. By Proposition 2.1.2 it suffices to show that $\eta\colon \oplus_i M_i \times_R N \to N$ is a surjection. Let $\pi_i\colon M_i \times_R N \to N$ be the natural map. Fix $x \in N$. Then $a_ig(x) \in a_iR = im(f_i)$ and $a_ig(x) \in im(g)$. Thus $a_i \cdot x \in im(\pi_i) \subset N$ for all $i \in I$. Therefore, $x = 1_R \cdot x$ is in $\oplus_i im(\pi_i) = im(\eta)$ since R is a unital ring and $(a_i \mid i \in I) = R$. $\square$

Proposition 6.0.6. Suppose $S = \langle\{f_1\colon M_1 \to R, f_2\colon M_2 \to R\}\rangle$ is a sieve on R such that $im(f_i) = a_i R$ for $i = 1, 2$. Then S is in the canonical topology on R-**Mod** if and only if $(a_1, a_2) = R$.

Proof. If S is in the canonical topology, then S is a colim sieve and hence by Proposition 6.0.4, $a_1 R \oplus a_2 R = R$.

If $(a_1, a_2) = R$, then by Proposition 6.0.4, S is a colim sieve. The universality of S follows immediately from Lemma 2.1.1, Proposition 6.0.4 and Lemma 6.0.5. □

Next we include two results that can help us identify when a sieve is not in the canonical topology.

Proposition 6.0.7. Let R be any nonzero ring. Let $S = \langle\{f_i\colon A_i \to X\}_{i\in I}\rangle$ be any sieve on X for any nonzero R-module X. If there exists a nonzero $b \in X$ such that $span_R(b) \subset (X - \cup_I Im(f_i)) \cup \{0\}$, then S is not a universal colim sieve.

Proof. Suppose such a $b \in X$ exists. Define $g\colon R \to X$ by $1 \to b$. Then $Im(g) \cap Im(f_i) = \{0\}$ for all i. Thus for all i, the pullback $R \times_X A_i = ker(g) \times ker(f_i)$ and the image of the natural map $R \times_X A_i \to R$ is $ker(g)$. In particular, $Im\left(\oplus_i R \times_X A_i \to R\right) = ker(g)$, which by construction is not R. Therefore, $\varinjlim_{g^*S} U \to R$ is not surjective and so g^*S not a colim sieve on R. □

Proposition 6.0.8. Let R be an infinite principal ideal domain. Let

$$S = \langle\{g_i\colon R^n \hookrightarrow R^n\}_{i=1}^{M} \cup \{f_i\colon R^{m_i} \hookrightarrow R^n \mid m_i < n\}_{i=1}^{N}\rangle$$

be a sieve on R^n. If S is a universal colim sieve, then $g_1 \oplus \cdots \oplus g_M\colon R^{nM} \to R^n$ is a surjection.

Proof. Let $G = g_1 \oplus \cdots \oplus g_M$. Suppose that G is not a surjection. We will produce a map ϕ that shows S is not universal.

By a change of basis (which is allowable by Lemma 2.1.3) we may assume that $G = diag(d_1, d_2, \ldots, d_n)$ with $d_i | d_{i+1}$. Because G is not surjectve, then d_n is not a unit. Indeed, if d_n was a unit, then all of the d_i's would also be units and thus G would be surjective. By Lemma 6.0.9 below, there exists an $x \in R^{n-1}$ so that $span_R\{(x,1)\} \cap Im(f_i) = \{0\}$ for all $i = 1, \ldots, N$. Additionally, since d_n is not a unit, then $(x, 1) \notin Im(G)$.

Define $\phi \colon R \to R^n$ by $1 \mapsto (x, 1)$. We will show that ϕ^*S is not a colim sieve. First we will simplify the generating set of ϕ^*S. By the choice of x, the pullback module of R^{m_i} along ϕ is $\{0\}$ for all $i = 1, \ldots, N$. Therefore, we can write ϕ^*S as $\phi^*S = \langle \{\pi_i \colon R^n \times_{R^n} R \to R\}_{i=1}^M \rangle$ where the π_i are the pullbacks of the g_i along ϕ. Since $(x, 1) \notin Im(G)$ and we have the following commutative diagram

$$\begin{array}{ccc}
\oplus_{i=1}^M R_i^n \times_{R^n} R & \xrightarrow{\oplus_{i=1}^M \pi_i} & R \\
\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\
\oplus_{i=1}^M R_i^n & \xrightarrow[G]{} & R^n
\end{array}$$

then $1 \notin Im(\pi_1 \oplus \cdots \oplus \pi_M)$. Therefore, $\eta \colon \underset{\overrightarrow{\phi^*S}}{\operatorname{colim}} U \to R$ is not surjective; hence ϕ^*S is not a colim sieve. $\square$

Lastly, for completeness we include the linear algebra result referenced in Proposition 6.0.8.

Lemma 6.0.9. Let R be an infinite principal ideal domain. For any finite collection $V_1, \ldots, V_N$ of submodules of R^n with $\dim(V_i) < n$, there exists an $x \in R^{n-1}$ such that $span_R\{(x, 1)\} \cap V_i = \{0\}$ for all i.

Proof. Let F be the quotient field of R. Let

$$W_i = V_i \otimes_R F.$$

We will use F^{n-1} to refer to the subspace $\{(a_1, \ldots, a_{n-1}, 0) \mid a_i \in F\}$ in F^n. For each $V_i \not\subset F^{n-1}$, fix an element $\nu_i \in V_i$ such that $\nu_i \notin F^{n-1}$ and write $\nu_i = (v_{i1}, \ldots, v_{in})$. Let $\nu_i^0 = (v_{i1}, \ldots, v_{i(n-1)}, 0)$. Lastly, for each $V_i \not\subset F^{n-1}$, define a vector space map $\phi_i \colon W_i \to F^{n-1}$ by $w = (w_1, \ldots, w_n) \mapsto w - \frac{w_n}{v_{in}}\nu_i$

Ideally, we will find an x such that $(x, 1) \notin W_i$ for all i. So first, let's see what kinds of $(z, 1)$ are in W_i by computing $\phi_i(z, 1)$.

$$\begin{aligned} \phi_i(z,1) &= (z,1) - \frac{1}{v_{in}}\nu_i \\ &= z - \frac{1}{v_{in}}\nu_i^0 \end{aligned}$$

Thus

$$z = \phi_i(z,1) + \frac{1}{v_{in}}\nu_i^0.$$

Therefore, if $(z, 1) \in W_i$, then $z = \phi_i(z, 1) + \frac{1}{v_{in}}\nu_i^0$. Based on this result, define $\Gamma_i = im(\phi_i) \oplus span_F\{\nu_i^0\}$. So $(z, 1) \in W_i$ implies $z \in \Gamma_i$.

For each index i exactly one of the following is true:

1. $W_i \subset F^{n-1}$,

2. $W_i \not\subset F^{n-1}$ and $\dim_F(\Gamma_i) < n - 1$,

3. $W_i \not\subset F^{n-1}$ and $\Gamma_i = F^{n-1}$.

For every index j in collection 1, every $x \in R^{n-1}$ satisfies the equation $span_R\{(x,1)\} \cap V_j = \{0\}$. Thus when picking our x, we only need to consider the indices in collections 2 and 3.

For each index i in collection 2, Γ_i is a proper subspace of F^{n-1}. Since there are only finitely many Γ_i and F is an infinite field, then there exists a $y = (y_1, \ldots, y_{n-1})$ such that $y \neq 0$ and $span_F\{(y,0)\} \cap \Gamma_i = \{0\}$ for all i in collection 2. By multiplying y by an appropriate $s \in F$ we can clear denominators and so we may assume that $y \in R^{n-1}$. In particular, for all $r \in R$, $ry \notin \Gamma_i$, which implies that $(ry,1) \notin W_i$. Therefore, for all $r \in R$, $span_R\{(ry,1)\} \cap V_i = \{0\}$ for all indices in collection 2.

Continuing with the y from the previous paragraph, we now consider the indices k in collection 3 and their corresponding Γ_k. In this situation, $(y,0) \in \Gamma_k$, i.e. $y = \phi_k(z) + u_k\nu_k^0$ for some $z \in W_k$ and $u_k \in F$. Since R is an infinite ring and collection 3 contains finitely many indices k, we can pick a nonzero $\rho \in R$ such that for all k, $\rho u_k \in R$ and $\rho u_k \neq \frac{1}{v_{kn}}$. Thus $\rho y \neq \phi_k(a) + \frac{1}{v_{kn}}\nu_k^0$ for any $a \in W_k$, which implies that $(\rho y, 1) \notin W_k$. Therefore, $span_R\{(\rho y,1)\} \cap V_k = \{0\}$ for all indices in collection 3.

We can take $x = \rho y$.

□

Examples

Here we include a few examples and non-examples of sieves in the canonical topology for various rings R.

Example 6.0.10. In the category of R-modules every surjective map generates a universal colim sieve (see Proposition 6.0.4). As more specific examples, the sieve

$\langle\{\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/n\mathbb{Z} \mid 1 \mapsto 1\}\rangle$ is in the canonical topology on **Ab** and in R-**Mod**, the sieve $\langle\{R^n \to R \mid (a_1, \ldots, a_n) \mapsto a_1\}\rangle$ is in the canonical topology.

Example 6.0.11. By Proposition 6.0.6, $\langle\{R \xrightarrow{a} R, R \xrightarrow{b} R\}\rangle$ is in the canonical topology if and only if $(a, b) = R$. As more specific examples, $\langle\{\mathbb{Z} \xrightarrow{2} \mathbb{Z}, \mathbb{Z} \xrightarrow{3} \mathbb{Z}\}\rangle$ is in the canonical topology on **Ab**; and when $\cdot g(x)\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is the map $f(x) \mapsto (g \cdot f)(x)$, $\langle\{C^\infty(\mathbb{R}) \xrightarrow{\cdot x} C^\infty(\mathbb{R}), C^\infty(\mathbb{R}) \xrightarrow{\cdot \sin(x)} C^\infty(\mathbb{R})\}\rangle$ is not in the canonical topology on $C^\infty(\mathbb{R})$-modules.

Example 6.0.12. The sieve $S = \langle\{R \xrightarrow{i_1} R^2, R \xrightarrow{i_2} R^2\}\rangle$ where $i_1(1) = (1, 0)$ and $i_2(1) = (0, 1)$ (in the category of R-modules for nontrivial R) is not in the canonical topology. By Proposition 6.0.4, S is clearly a colim sieve so to see that S is not universal consider the map $\Delta\colon R \to R^2$, $1 \mapsto (1, 1)$. Then for $k = 1, 2$, i_k pulled back along Δ yields the zero map $z\colon 0 \to R$. Hence Lemma 2.1.1 says $\Delta^* S = \langle\{z\colon 0 \to R\}\rangle$, which is clearly not a colim sieve.

Similarly $\langle\{R \xrightarrow{i_k} R^n \mid k = 1, \ldots, n\}\rangle$ is a colim sieve but is not in the canonical topology. (This is also a consequence of Proposition 6.0.7.)

Example 6.0.13. Let $S = \langle\{f_k\colon \mathbb{Q} \to \mathbb{Q}[t] \mid f_k(1) = 1 + t + \cdots + t^k\}_{k=1}^\infty\rangle$ in the category of rational vector spaces. This S is not in the canonical topology. (This is a direct consequence of Proposition 6.0.7 using $b = t$.)

Example 6.0.14. Let F be an infinite field. In the category of F vector spaces, a sieve of the form $S = \langle\{F^{m_i} \hookrightarrow F^n \mid m_i \leq n\}_{i=1}^M\rangle$ is in the canonical topology if and only if $m_i = n$ for some i if and only if S contains an isomorphism. (This is a consequence of Proposition 6.0.8.)

Proposition 6.0.15. Consider the diagram $B_1 \hookrightarrow B_2 \hookrightarrow B_3 \hookrightarrow \ldots$ made with only injective maps and the direct limit $B := \varinjlim B_n$ in R-**mod**. Let $\iota_n\colon B_n \to B$ be the natural maps into the colimit. Then the sieve $\langle\{\iota_n \,|\, n \in \mathbb{N}\}\rangle$ is a universal colim sieve.

Proof. Let $\Gamma\colon \mathbb{N} \to S$ by $n \mapsto \iota_n$. Notice that Γ is a final functor; this is easy to see since the injectivity of ι_n and the maps in our diagram imply that $B_i \times_B B_j \cong B_{min(i,j)}$. Thus $\varinjlim_S U$ exists and $\varinjlim_S U \cong \varinjlim_{\mathbb{N}} U\Gamma \cong B$. Therefore, S is a colim sieve.

To see that S is universal, let $f\colon X \to B$ and set $X_i \coloneqq X \times_B B_i$. For each $n \in \mathbb{N}$, ι_n and $B_n \to B_{n+1}$ are both injective maps; this implies that the natural maps $X_n \to X_{n+1}$ and $X_n \to X$ are also injective maps since the pullback of an injection in R-**Mod** is an injection and $X_i \cong X_{i+1} \times_{B_{i+1}} B_i$. Additionally, it is an easy exercise to see that the direct limit $\varinjlim X_i$ is isomorphic to X. In other words, f^*S is the type of sieve described in the assumptions of this proposition and proved to be a colim sieve in the previous paragraph. $\square$

Example 6.0.16. Take $B_n = \mathbb{R}^n$ and let $B_n \to B_{n+1}$ be the inclusion map $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0)$. Use $\mathbb{R}^\infty$ to denote the direct limit. Then the above proposition shows that $\langle\{\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty\}_{n\in\mathbb{N}}\rangle$ is in the canonical topology on the category of $\mathbb{R}$ vector spaces. (Compare this to Example 5.2.7.)

Reductions

In this part we prove some reductions that allow us to limit our view (of sieve generating sets and the maps universality must be checked over) to the non-full subcategory of free modules with injective maps when R is 'nice.' The first reduction will be reducing the types of sieves we need to look at:

Proposition 6.0.17 (Reduction 1)**.** In R-**Mod**, let S be a sieve on X. Then the following are equivalent

1. S is a universal colim sieve

2. f^*S is a universal colim sieve for every surjection $f\colon Y \to X$

3. f^*S is a universal colim sieve for some surjection $f\colon Y \to X$

Proof. It is obvious that 1 implies 2 and 2 implies 3, so it suffices to show 3 implies 1.

Assume f^*S is a universal colim sieve for some fixed surjection $f\colon Y \to X$. Set $T = \langle\{f\colon Y \to X\}\rangle$. By Proposition 6.0.4, T is a universal colim sieve since f is a surjection. We will now use T together with the Grothendieck topology's transitivity axiom to show that S is a universal colim sieve. Notice that S satisfies the hypotheses of this axiom with respect to T. Indeed, since every $(g\colon Z \to X) \in T$ factors as $f \circ k\colon Z \to Y \to X$ for some k, then $g^*S = (fk)^*S = k^*(f^*S)$, which implies that g^*S is a universal colim sieve (as f^*S is universal) for every $g \in T$. Therefore, by the transitivity axiom of a Grothendieck topology, S is a universal colim sieve. $\square$

To rephrase our first reduction: S is a universal colim sieve on X if and only if f^*S is a universal colim on R^n where $f\colon R^n \to X$ is a surjection (note that n is not necessarily assumed to be finite). This reduction means that we can restrict our view to free modules (not necessarily finitely generated). Specifically, we only need to look at sieves on free modules and check the universality condition on free modules. Indeed, S is a universal colim sieve on X if and only if for all $g\colon Y \to X$, g^*S is a universal colim sieve on Y if and only if for all $g\colon Y \to X$, $(gf)^*S$ is a universal colim sieve on R^n for some surjection $f\colon R^n \to Y$.

Proposition 6.0.18 (Reduction 2)**.** In R-**Mod** when R is a principal ideal domain, every sieve on R^n equals a sieve of the form

$$\langle\{g_i\colon R^{m_i} \hookrightarrow R^n : m_i \leq n\}_{i\in I}\rangle$$

where the g_i are injections.

Proof. Let $S = \langle\{f_i\colon A_i \to R^n\}_{i\in I}\rangle$ be a sieve on R^n. Set $T = \langle\{g_i\colon Im(f_i) \to R^n\}_{i\in I}\rangle$ where the g_i's are inclusion maps. Since R is a PID and $Im(f_i)$ is a submodule of R^n, then $Im(f_i) \cong R^{m_i}$ for some $m_i \leq n$. Thus T is of the desired form and we will show that $S = T$. First notice that $S \subset T$. To get that T is a subcollection of S, notice that $\tilde{f}_i\colon A_i \to Im(f_i)$ (i.e. f_i with a different codomain) is split because $\tilde{f}_i$ is a surjective map onto a projective module; call the splitting χ_i. Hence $g_i = g_i \circ \tilde{f}_i \circ \chi_i = f_i \circ \chi_i$ implies that $T \subset S$ and completes the proof. □

To rephrase our second reduction: when talking about sieves on R^n, we only need to talk about sieves generated by injections of free modules. Thus we can restrict our view of sieve generating sets to the non-full subcategory of free modules with injective morphisms.

Our next reduction will also assume R is a principal ideal domain. In particular, fix n and a map $f\colon X \to R^n$ for some R-module X. Then since R is a PID, we may write

$$X \cong R^m \oplus K \quad \text{for some } m \leq n\text{, where}$$

$$R^m \cong Im(f), \quad K = ker(f), \quad f = g + z \quad \text{with}$$

$$g\colon R^m \to R^n \text{ an injection and } z\colon K \to R^n \text{ the zero map.}$$

Proposition 6.0.19 (Reduction 3)**.** Let R be a principal ideal domain, S be a sieve on R^n in R-**Mod** and $f\colon X \to R^n$. Then, using the set-up described in the previous paragraph,

$$\varinjlim_{f^*S} U \cong \left(\varinjlim_{g^*S} U\right) \oplus \left(\varinjlim_{z^*S} U\right).$$

Moreover, z^*S is a universal colim sieve; hence f^*S is a colim sieve if and only if g^*S is a colim sieve.

Sketch of Proof. By Proposition 6.0.18, we may assume that S can be written in the form $S = \langle\{\eta_i\colon R^{p_i} \hookrightarrow R^n : p_i \leq n\}_{i\in I}\rangle$. Consider the diagrams $\mathcal{X}$, $\mathcal{R}$ and $\mathcal{K}$ defined as:

$$\mathcal{X} = \left(\begin{array}{c} \bigoplus_{i\in I}(R^{p_i} \times_{R^n} X) \times_X (R^{p_i} \times_{R^n} X) \\ \downarrow\downarrow \\ \bigoplus_{i\in I}(R^{p_i} \times_{R^n} X) \end{array}\right),$$

$$\mathcal{R} = \left(\begin{array}{c} \bigoplus_{i\in I}(R^{p_i} \times_{R^n} R^m) \times_{R^m} (R^{p_i} \times_{R^n} R^m) \\ \downarrow\downarrow \\ \bigoplus_{i\in I}(R^{p_i} \times_{R^n} R^m) \end{array}\right), \text{ and}$$

$$\mathcal{K} = \left(\begin{array}{c} \bigoplus_{i\in I}(R^{p_i} \times_{R^n} K) \times_K (R^{p_i} \times_{R^n} K) \\ \downarrow\downarrow \\ \bigoplus_{i\in I}(R^{p_i} \times_{R^n} K) \end{array}\right)$$

First we look at the objects of $\mathcal{X}$. Since each η_i is injective, then for all i

$$R^{p_i} \times_{R^n} X \cong (R^{p_i} \times_{R^n} R^m) \oplus (R^{p_i} \times_{R^n} K)$$

and for all i, q

$$\begin{aligned}&(R^{p_i} \times_{R^n} X) \times_X (R^{p_q} \times_{R^n} X)\\ &\qquad \cong ((R^{p_i} \times_{R^n} R^m) \times_{R^m} (R^{p_q} \times_{R^n} R^m)) \oplus ((R^{p_i} \times_{R^n} K) \times_K (R^{p_q} \times_{R^n} K)).\end{aligned}$$

In other words, $\mathcal{X} \cong \mathcal{R} \oplus \mathcal{K}$. But since colimits "commute" with colimits, then $\mathrm{Coeq}(\mathcal{X}) \cong \mathrm{Coeq}(\mathcal{R}) \oplus \mathrm{Coeq}(\mathcal{K})$. Now by Lemma 2.1.1 and Proposition 2.1.2, the first part has been proven, i.e.

$$\varinjlim_{f^*S} U \cong \left(\varinjlim_{g^*S} U\right) \oplus \left(\varinjlim_{z^*S} U\right).$$

Next we notice that z^*S is a universal colim sieve. Indeed, since η_i is an injection and z is the zero map, it easily follows that $z^*S = \langle\{id\colon K \to K\}\rangle$.

To complete the proof, notice that we have the following commutative diagram

$$\begin{tikzcd}[column sep=tiny] \mathrm{Coeq}(\mathcal{X}) \arrow[dr, bend right, "\chi"] & \cong & \mathrm{Coeq}(\mathcal{R}) \arrow[d, "\rho"] & \oplus & \mathrm{Coeq}(\mathcal{K}) \arrow[dl, bend left, "\kappa"] \\ & X & \cong R^m & \oplus K & \end{tikzcd}$$

where the vertical maps are the obvious canonical maps. This $\chi = \rho \oplus \kappa$ is an isomorphism if and only if both ρ and κ are isomorphisms. We have already shown that κ is an isomorphism (as z^*S is a universal colim sieve), thus this diagram implies that χ is an isomorphism if and only if ρ is; hence f^*S is colim sieve if and only if g^*S is a colim sieve.

□

Lastly, we rephrase our third reduction:

Corollary 6.0.20. When R is a PID, a sieve on R^n is a universal colim sieve if and only if f^*S is a colim sieve for every injection $f\colon R^m \to R^n$.

All together our reductions basically allow us to work in the subcategory of free modules with injective morphisms instead of in R-**Mod**.

6.1 The Category of Abelian Groups

This section will be primarily made up of examples. Additionally, we include a characterization of sieves on $\mathbb{Z}$ and one result for sieves on larger free abelian groups.

Example 6.1.1. By Corollary 6.0.6, $\langle\{\mathbb{Z} \xrightarrow{\times a} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times b} \mathbb{Z}\}\rangle$ is a universal colim sieve if and only if a and b are relatively prime.

Example 6.1.2. The sieve $S = \langle\{\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z}\}\rangle$ is a universal colim sieve on $\mathbb{Z}/4\mathbb{Z}$ by Corollary 6.0.2. Additionally, S is not monogenic, i.e. it cannot be written as a sieve generated by one morphism.

Example 6.1.3. Let $S = \langle\{g\colon \mathbb{Z}^n \hookrightarrow \mathbb{Z}^n\} \cup \{f_i\colon \mathbb{Z}^{m_i} \hookrightarrow \mathbb{Z}^n \mid m_i < n\}_{i=1}^N\rangle$ be a sieve on $\mathbb{Z}^n$. Then S is a universal colim sieve if and only if g is a surjection, i.e. g is an isomorphism. (This is a direct corollary of Proposition 6.0.8 and Corollary 6.0.2.)

Ideally, we would like to know a 'nice' basis for the canonical topology on **Ab**, like the bases in Section 5.1; to start moving towards this ideal, we look at the simplest free group, $\mathbb{Z}$. In Example 6.1.1 we see that a relative prime pair of numbers will generate a universal colim sieve; this is actually true in general, specifically:

Proposition 6.1.4. Let $S = \langle\{\mathbb{Z} \xrightarrow{\times a_i} \mathbb{Z}\}_{i=1}^N\rangle$ be a sieve on $\mathbb{Z}$. Then S is a universal colim sieve if and only if $\gcd(a_1, \ldots, a_N) = 1$.

Proof. First assume that S is a universal colim sieve. In particular, $\varinjlim_S U \to \mathbb{Z}$ is a surjection, i.e. $\mathbb{Z}^N \to \mathbb{Z}$, $(x_1, \ldots, x_N) \mapsto a_1x_1 + \cdots + a_Nx_N$ is a surjection. Therefore, $(a_1, \ldots, a_N) = \mathbb{Z}$ and this proves the forward direction.

Now assume that $\gcd(a_1, \ldots, a_N) = 1$. We will break the proof that S is a universal colim sieve up into several pieces. First we will reduce the proof to showing that S is a colim sieve. By the reductions (Propositions 6.0.17, 6.0.18 and 6.0.19), universality only needs to be checked along maps of the form $f\colon \mathbb{Z} \xrightarrow{\times k} \mathbb{Z}$ where $k \neq 0$. Fix $k \neq 0$, i.e. fix f, and write $\mathbb{Z}_b$ for the domain of $\mathbb{Z} \xrightarrow{\times b} \mathbb{Z}$. By Lemma 2.1.1, $f^*S = \langle \{\pi_i\colon \mathbb{Z}_{a_i} \times_{\mathbb{Z}} \mathbb{Z}_k \to \mathbb{Z}_k\}_{i=1}^N \rangle$. Moreover, it is easy to see that the pullback $\mathbb{Z}_{a_i} \times_{\mathbb{Z}} \mathbb{Z}_k \cong \mathbb{Z}$ and π_i must be multiplication by $\frac{a_i}{\gcd(a_i,k)}$. Since $\gcd(a_1, \ldots, a_N)$ equals 1, then $\gcd\left(\frac{a_1}{\gcd(a_1,k)}, \ldots, \frac{a_N}{\gcd(a_N,k)}\right) = 1$ and hence f^*S has the same form as S. Specifically, any argument showing that S is a colim sieve will similarly show that f^*S is a colim sieve. Therefore, it suffices to show that S is a colim sieve.

To see that S is a colim sieve, i.e. to see that the map $\varinjlim_S U \to \mathbb{Z}$ induced by $a_1, \ldots, a_N$ is an isomorphism, let $\alpha = \frac{N(N-1)}{2}$ and notice that

$$\begin{aligned}
\varinjlim_S U &\cong \operatorname{Coeq}\left(\begin{array}{c} \oplus_{i=1}^{\alpha} \mathbb{Z} \\ \downarrow\;\downarrow \\ \oplus_{i=1}^{N} \mathbb{Z} \end{array}\right) \\
&\cong \operatorname{Cokernel}\left(\phi\colon \mathbb{Z}^{\alpha} \to \mathbb{Z}^N\right)
\end{aligned}$$

for some map ϕ where the first isomorphism comes from Lemma 2.1.2 and the last isomorphism comes from the fact that we are working in an abelian category. Now this map ϕ happens to be the third map in the Taylor resolution of $\mathbb{Z}$, i.e. ϕ_1 in [12]. We make two remarks about this previous sentence: (1) we will not prove that our ϕ is [12]'s ϕ_1, although this is easy to observe, and (2) the Taylor resolution in [12] is specifically for polynomial rings, not $\mathbb{Z}$, however, both the definition of the Taylor resolution and the proof that it is in fact a free resolution are analogous. Here is the end of the Taylor resolution:

$$\cdots \to \mathbb{Z}^\alpha \xrightarrow{\phi} \mathbb{Z}^N \xrightarrow{(a_1 \ \dots \ a_N)} \mathbb{Z} \to \mathbb{Z}/(a_1, \dots, a_N)\mathbb{Z} \to 0$$

Since $\gcd(a_1, \dots, a_N) = 1$, then it follows that $(a_1 \ \dots \ a_N)$ is a surjection and $\mathbb{Z}/(a_1, \dots, a_N)\mathbb{Z} \cong 0$. Thus we obtain an exact sequence $0 \to Im(\phi) \to \mathbb{Z}^N \to \mathbb{Z} \to 0$, which implies that the cokernel of ϕ is $\mathbb{Z}$. Additionally, since $(a_1 \ \dots \ a_N)$ induced our map $\varinjlim_S U \to \mathbb{Z}$, then this short exact sequence also says that S is a colim sieve. $\square$

Because of Proposition 6.1.4, we can now easily determine when a sieve on $\mathbb{Z}$ is in the canonical topology and we can easily come up with examples; for example, $\langle\{\mathbb{Z} \xrightarrow{\times 15} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 10} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 12} \mathbb{Z}\}\rangle$ is in the canonical topology whereas the sieve $\langle\{\mathbb{Z} \xrightarrow{\times 15} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 50} \mathbb{Z}, \mathbb{Z} \xrightarrow{\times 20} \mathbb{Z}\}\rangle$ is not. One may hope for a similar outcome for sieves on $\mathbb{Z}^n$ when $n \geq 2$, however, the Taylor resolution used in the proof of Proposition 6.1.4 does not seem to generalize in a suitable manner. Instead, we have a proposition that may tell us when a potential sieve is not in the canonical topology.

Proposition 6.1.5. Let $S = \langle\{\mathbb{Z}^n \xrightarrow{A_i} \mathbb{Z}^n\}_{i=1}^N\rangle$ where A_i is a diagonal matrix with $\det(A_i) \neq 0$. Then there exists a map $\beta\colon \mathbb{Z} \to \mathbb{Z}^n$ such that $\beta^* S$ is not a colim sieve if and only if $\gcd(\det(A_1), \dots, \det(A_N)) \neq 1$.

Proof. First we set up some notation: Let $A_i = diag(a_{1i}, \dots, a_{ni})$ and $\mathbb{Z}^n_i$ be the domain of A_i.

To prove the backward direction, suppose that $\gcd(\det(A_1), \dots, \det(A_N)) \neq 1$. We can rephrase the assumptions as $a_{ik} \neq 0$ for all k and there exists a prime q such that q divides the product $a_{1i} \dots a_{ni}$ for all i. Set β equal to the diagonal embedding, i.e. $1 \mapsto (1, \dots, 1)$. By Lemma 2.1.1, $\beta^* S = \langle \{f_i \colon \mathbb{Z}_i^n \times_{\mathbb{Z}^n} \mathbb{Z} \to \mathbb{Z}\}_{i=1}^N \rangle$. Let $k_i = \mathrm{lcm}(a_{1i}, \dots, a_{ni})$ and $\chi_i \colon \mathbb{Z} \to \mathbb{Z}^n$, $1 \mapsto \left(\frac{k_i}{a_{1i}}, \dots, \frac{k_i}{a_{ni}}\right)$, then

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\chi_i"] \arrow[d, "k_i"'] & \mathbb{Z}^n \arrow[d, "A_i"] \\ \mathbb{Z} \arrow[r, "\beta"'] & \mathbb{Z}^n \end{tikzcd}$$

is a pullback diagram. Moreover, the prime q divides k_i for all i since it divides $a_{1i} \dots a_{ni}$ for all i. Thus $\gcd(k_1, \dots, k_N) \neq 1$. Now by Proposition 6.1.4, we can see that $\beta^* S = \langle \{\mathbb{Z} \xrightarrow{\times k_i} \mathbb{Z}\}_{i=1}^N \rangle$ is not a universal colim sieve. In particular, the first part of the proof of Proposition 6.1.4 shows that $\beta^* S$ is not a colim sieve.

To prove the forward direction, we will prove the contrapositive statement. So suppose that $\gcd(\det(A_1), \dots, \det(A_N)) = 1$. Let $\beta \colon \mathbb{Z} \to \mathbb{Z}^n$ be given as the matrix $\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. To see that $\beta^* S = \langle \{f_i \colon \mathbb{Z}_i^n \times_{\mathbb{Z}^n} \mathbb{Z} \to \mathbb{Z}\}_{i=1}^N \rangle$ is a colim sieve, notice that we have the pullback diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d, "k_i"'] & \mathbb{Z}^n \arrow[d, "A_i"] \\ \mathbb{Z} \arrow[r, "\beta"'] & \mathbb{Z}^n \end{tikzcd}$$

where $k_i = \mathrm{lcm}\left(\frac{a_{1i}}{\gcd(a_{1i}, b_1)}, \dots, \frac{a_{ni}}{\gcd(a_{ni}, b_n)}\right)$. Clearly, k_i divides $\det(A_i) = a_{1n} \dots a_{ni}$. This implies that $\gcd(k_1, \dots, k_n)$ divides $\gcd(\det(A_1), \dots, \det(A_N))$ and hence equals 1. Now by Proposition 6.1.4, we can see that $\beta^* S = \langle \{\mathbb{Z} \xrightarrow{\times k_i} \mathbb{Z}\}_{i=1}^N \rangle$ is a universal colim sieve. $\square$

Example 6.1.6. Based on Proposition 6.1.5 we can say automatically that the sieve $\left\langle \left\{ \begin{pmatrix} 4 & 0 \\ 0 & 14 \end{pmatrix}, \begin{pmatrix} 21 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 49 \end{pmatrix} \right\} \right\rangle$ on $\mathbb{Z}^2$ is not in the canonical topology because each matrix has a multiple of 7 somewhere on its diagonal.

Suppose, like in Proposition 6.1.5, $S = \langle\{\mathbb{Z}^n \xrightarrow{A_i} \mathbb{Z}^n\}_{i=1}^N\rangle$ where each A_i is a diagonal matrix and $\gcd(\det(A_1),\ldots,\det(A_N)) = 1$. In order to determine if S is a universal colim sieve, we (only) need to check if f^*S is a colim sieve for all $f\colon \mathbb{Z}^m \hookrightarrow \mathbb{Z}^n$, $2 \le m \le n$. However, this is still a fair amount of work and it would be nice if this process could be simplified further.

Now we finish this section with a few more examples. Note: we will not prove any assertions in these examples, however, they are all basic computations that can be checked using undergraduate linear algebra.

Example 6.1.7. The sieve $S_1 = \left\langle\left\{\begin{pmatrix}7 & 0\\ 1 & 4\end{pmatrix}, \begin{pmatrix}21 & 0\\ 1 & 18\end{pmatrix}, \begin{pmatrix}24 & 0\\ 6 & 5\end{pmatrix}\right\}\right\rangle$ on $\mathbb{Z}^2$ is not in the canonical topology although it is a colim sieve. In particular, S_1 is not universal because f^*S_1 is not a colim sieve for $f\colon \mathbb{Z} \to \mathbb{Z}^2$, $f(1) = (1,0)$.

If we take the generating set of S_1 and change the 1 in the first matrix to a 0, then we get the following example:

Example 6.1.8. The sieve $S_2 = \left\langle\left\{\begin{pmatrix}7 & 0\\ 0 & 4\end{pmatrix}, \begin{pmatrix}21 & 0\\ 1 & 18\end{pmatrix}, \begin{pmatrix}24 & 0\\ 6 & 5\end{pmatrix}\right\}\right\rangle$ on $\mathbb{Z}^2$ is not a colim sieve since $\varinjlim_S U \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$. Therefore, S_2 is also not in the canonical topology.

Finally, if take the generating set of S_2 and change the 18 in the second matrix to a 9, then we get:

Example 6.1.9. The sieve $S_3 = \left\langle\left\{\begin{pmatrix}7 & 0\\ 0 & 4\end{pmatrix}, \begin{pmatrix}21 & 0\\ 1 & 9\end{pmatrix}, \begin{pmatrix}24 & 0\\ 6 & 5\end{pmatrix}\right\}\right\rangle$ on $\mathbb{Z}^2$ is a colim sieve, however, whether or not this sieve is in the canonical topology is unknown.

We find these last three examples particularly interesting because it helps showcase how small tweaks in our generating set can led to different results.

CHAPTER VII

THE QUILLEN TOPOLOGY

In Chapter II page 5.5 of [14], Quillen introduces a very specific Grothendieck topology, which we dub *the Quillen topology.* His definition yields something very similar to the canonical topology, but (as we will explain) is ultimately different. Additionally, Quillen's topology on the category of R-modules leads to a nice connection with Ext, and we raise the question of whether a similar result works for the canonical topology. In this section we will talk a little bit about this topology; specifically, how it relates to the canonical topology and how, when applied to the category of R-modules, it relates to Ext. Quillen defines his topology by declaring a Grothendieck basis, which we will rephrase as:

Definition 7.0.1. For a category $\mathcal{C}$ and object X of $\mathcal{C}$, set $\mathcal{Q}(X)$ to be the collection of sets of the form $\{f\}$ where f is a universal effective epimorphism with codomain X.

Remark 7.0.2. In [14], Quillen additionally assumes that effective epimorphisms in $\mathcal{C}$ are always universal; however, we omit this assumption so that we can work in more generality.

We will now show that this collection of universal effective epimorphisms forms a Grothendieck basis:

Proposition 7.0.3. The function $\mathcal{Q}$ that assigns each object X of category $\mathcal{C}$ the collection $\mathcal{Q}(X)$, is a Grothendieck basis whenever $\mathcal{C}$ has all pullbacks.

Proof. We only need to show three things:

1. if $f\colon Y \to X$ is an isomorphism, then $\{f\} \in \mathcal{Q}(X)$,

2. if $\{f\colon Y \to X\} \in \mathcal{Q}(X)$ and $g\colon Z \to X$ is any morphism, then $\{\pi_2\colon Y \times_X Z \to Z\} \in \mathcal{Q}(Z)$,

3. if $\{f\colon Y \to X\} \in \mathcal{Q}(X)$ and $\{g\colon Z \to Y\} \in \mathcal{Q}(Y)$, then $\{f \circ g\colon Z \to X\} \in \mathcal{Q}(X)$.

The first and second conditions are automatically satisfied. The third condition follows immediately from Corollary 2.1.7. □

Now Proposition 7.0.3 completely justifies the following definition:

Definition 7.0.4. Let $\mathcal{C}$ be a category with all pullbacks. The *Quillen topology* on $\mathcal{C}$ is the Grothendieck topology defined by: a sieve S on X is a cover if and only if S contains a universal effective epimorphism whose codomain is X.

Thus the Quillen topology is subcanonical, i.e. is contained in the canonical topology. Indeed, if S contains a universal effective epimorphism $f\colon A \to X$, then by Corollary 2.1.5, $\langle\{f\}\rangle$ is in the canonical topology and so S is in the canonical topology (as $\langle\{f\}\rangle \subset S$). However, the Quillen topology is not equal to the canonical topology; we can see this via the following example:

Example 7.0.5. In the category of sets, pick any two nonempty sets U and V. Consider the sieve $S = \langle\{i_1\colon U \to U \coprod V, i_2\colon V \to U \coprod V\}\rangle$ (where both maps are the usual inclusions into the coproduct). By Proposition 5.1.1, S is in the canonical topology since $i_1 + i_2 = id$ is a surjection. On the other hand, S is not in the Quillen topology because the universal effective epimorphisms in **Sets** are precisely surjections but S cannot contain a surjection since neither i_1 nor i_2 is a surjection.

Remark 7.0.6. By Corollary 2.1.5, every sieve in the canonical topology generated by one function is generated by a universal effective epimorphism. Therefore, the Quillen topology and canonical topology agree on all monogenic sieves.

Now we include a fun little corollary of [14]. Specifically, it expresses Ext as a sheaf cohomology. We note that although Quillen does not outright state this result, it is an immediate consequence of the double complex in Chapter II page 5.13 of [14] when we take our category to be R-modules. (Again, we assume that R is a commutative unital ring.)

Corollary 7.0.7. Let $\mathrm{H}^*_{\mathcal{Q}}$ be the sheaf cohomology on the category of R-modules with the Quillen topology. For any two R-modules N and M,

$$\mathrm{Ext}^*(N, M) = \mathrm{H}^*_{\mathcal{Q}}(N, rM).$$

Lastly, we raise the question of whether or not this (or a similar) result holds for the canonical topology. In general, what is the connection between the sheaf cohomology in the Quillen topology and the sheaf cohomology in the canonical topology?

CHAPTER VIII

THE HOMOTOPICAL CANONICAL TOPOLOGY

In this section we discuss the homotopical versions of (universal) colim sieves and the canonical Grothendieck topology.

Definition 8.0.1. For a model category $\mathcal{M}$, an object X of $\mathcal{M}$ and sieve S on X, we call S a *hocolim sieve* if the canonical map $\mathrm{hocolim}_S U \to X$ is a weak equivalence. Moreover, we call S a *universal hocolim sieve* if for all arrows $\alpha\colon Y \to X$ in $\mathcal{C}$, $\alpha^* S$ is a hocolim sieve.

Theorem 8.0.2. For a simplicial model category $\mathcal{M}$, the collection of all universal hocolim sieves on $\mathcal{M}$ forms a Grothendieck topology.

Definition 8.0.3. For a simplicial model category $\mathcal{M}$, the collection of all universal hocolim sieves on $\mathcal{M}$ is called the *homotopical canonical topology* on $\mathcal{M}$.

Our goal in this section is to prove Theorem 8.0.2, and thereby justify Definition 8.0.3. As such we will start with a discussion of the key parts of the proof and then develop the necessary tools.

Let $\mathcal{U}$ be the collection of universal hocolim sieves for the simplicial model category $\mathcal{M}$ with $\mathcal{U}(X)$ the collection of universal hocolim sieves on X. In order for the collection of all universal hocolim sieves to form a topology, for each object X in $\mathcal{M}$, there are three things that must be true:

1. $(\mathcal{M} \downarrow X) \in \mathcal{U}(X)$,
2. if $S \in \mathcal{U}(X)$ and $g\colon Y \to X$, then $g^* S \in \mathcal{U}(Y)$, and

3. (Transitivity) if $S \in \mathcal{U}(X)$ and R is any sieve on X such that for all $h \in S$, $h^*R \in \mathcal{U}(\text{domain}\, h)$, then $R \in \mathcal{U}(X)$.

As we will see later in the proof of Theorem 8.0.2, the first two of these requirements will be easy to obtain. Thus we will focus our discussion on the transitivity axiom. For the rest of the discussion, we fix a sieve $S \in \mathcal{U}(X)$ and a sieve R on X such that for all $f \in S$, $f^*R \in \mathcal{U}(\text{domain } f)$. For simplicity we will focus on showing that R is a hocolim sieve on X.

Remark 8.0.4. Throughout this section we will be using notation and definitions from Section 2.3.

Notation 8.0.5. We will use ${}_X[\,]$ for the subcategory of $(\mathcal{M} \downarrow X)$ containing $(id_X \colon X \to X)$ as its only object and no non-identity morphisms.

I. Basic Argument and Outline

We will be using the following noncommutative diagram of categories:

$$\begin{tikzcd} {}_X[R] \arrow[r, "\mathscr{F}"] & {}_X[\,] \\ {}_X[RSR] \arrow[u, "\mathscr{F}\circ\mathscr{F}"] \arrow[r, "\mathscr{C}"'] & {}_X[SR]. \arrow[u, "\mathscr{F}\circ\mathscr{F}"'] \arrow[ul, "\mathscr{C}"'] \end{tikzcd} \tag{8.1}$$

There are two important things to notice about this diagram:

1. The upper right triangle commutes. Indeed, ${}_X[\,]$ is the unique terminal category and thus both of the functors $\mathscr{F} \circ \mathscr{F}$ and $\mathscr{F} \circ \mathscr{C}$ must be equal.

2. The lower left triangle does not commute. Indeed, $\mathscr{F}^2(\rho, \tau, \gamma) = \rho$ whereas $\mathscr{C}^2(\rho, \tau, \gamma) = \rho \circ \tau \circ \gamma$. Instead there is a natural transformation $\eta \colon (\mathscr{C} \circ \mathscr{C}) \to (\mathscr{F} \circ \mathscr{F})$ given by $\eta_{(\rho,\tau,\gamma)} = \tau \circ \gamma$. Pictorially, for $X \xrightarrow{\rho} A \xrightarrow{\tau} B \xrightarrow{\gamma} C$,

$$\begin{array}{ccc} \mathscr{C}^2(\rho,\tau,\gamma) & & \\ {\scriptstyle \eta_{(\rho,\tau,\gamma)}}\downarrow & \text{is} & \begin{tikzcd} X \arrow[d, "id"'] & C \arrow[l, "\rho\circ\tau\circ\gamma"'] \arrow[d, "\tau\circ\gamma"] \\ X & A. \arrow[l, "\rho"] \end{tikzcd} \\ \mathscr{F}^2(\rho,\tau,\gamma) & & \end{array}$$

It is easy to see that η is indeed a natural transformation and so we will not show it here.

Now take diagram (8.1), thinking of each category ${}_X[T_1 \dots T_n]$ as the diagram $U\colon {}_X[T_1 \dots T_n] \to \mathcal{M}$, we apply homotopy colimits levelwise to get the following induced noncommutative diagram (with a commutative upper right triangle):

$$\begin{tikzcd} \mathrm{hocolim}_{{}_X[R]} U \arrow[r, "\mathscr{F}_*"] & \mathrm{hocolim}_{{}_X[\,]} U \arrow[r, "\simeq"] & X \\ \mathrm{hocolim}_{{}_X[RSR]} U \arrow[u, "(\mathscr{F}\circ\mathscr{F})_*"] \arrow[r, "\mathscr{C}_*"'] & \mathrm{hocolim}_{{}_X[SR]} U. \arrow[u, "(\mathscr{F}\circ\mathscr{F})_*"'] \arrow[ul, "\mathscr{C}_*"'] & \end{tikzcd} \tag{8.2}$$

Note: because ${}_X[\,]$ only has one object, namely id_X, then it's immediate that $\mathrm{hocolim}_{{}_X[\,]} U \simeq U(id_X) = X$.

Since ${}_X[R] = R$, then we can prove that R is a hocolim sieve on X by showing that the top horizontal map $\mathscr{F}_*$ in (8.2) is a weak equivalence. We outline the proof of this:

- First we show that all vertical maps $(\mathscr{F} \circ \mathscr{F})_*$ in (8.2) are weak equivalences.
- Then we prove that the lower left triangle in diagram (8.2) commutes up to homotopy.
  - ∗ We show that the natural transformation η mentioned with respect to diagram (8.1) gives a "homotopy" at the categorical level.
  - ∗ Then we set some notation and prove a lemma in order to get a convenient cylinder object for $|X|$ whenever X is 'nice' a simplicial object of $\mathcal{M}$.

    * Finally we use our cylinder object to show that the "categorical homotopy" induces a weak equivalence, i.e. in the homotopy category diagram (8.2) is commutative.

- Then it will follow formally that $\mathscr{F}_*\colon \operatorname{hocolim}_{{}_X[R]}U \to \operatorname{hocolim}_{{}_X[\ ]}U$ is a weak equivalence.

We will discuss the first two steps now and the third in the proof of Theorem 8.0.2.

II. Vertical Maps

We will show that all vertical maps $(\mathscr{F} \circ \mathscr{F})_*$ in (8.2) are weak equivalences. Since $(\mathscr{F} \circ \mathscr{F})_* = \mathscr{F}_* \circ \mathscr{F}_*$, then the first goal is completed by the following lemma:

Lemma 8.0.6. Let $n \geq 1$ and $T_1, \ldots, T_n$ be sieves on X such that for all $f \in T_{n-1}$, f^*T_n is a universal hocolim sieve. Then the induced map

$$\mathscr{F}_*\colon \operatorname{hocolim}_{{}_X[T_1 \ldots T_n]}U \to \operatorname{hocolim}_{{}_X[T_1 \ldots T_{n-1}]}U$$

is a weak equivalence. Note: when $n = 1$, then $T_{n-1} = \{id_X\colon X \to X\}$ and ${}_X[T_1 \ldots T_{n-1}] = {}_X[\ ]$.

Proof. We will use ρ as an abbreviation for $(\rho_1, \ldots, \rho_{n-1}) \in {}_X[T_1 \ldots T_{n-1}]$. Additionally, we will abuse notation and use ρ to represent $\rho_1 \circ \cdots \circ \rho_{n-1}$ (e.g. ρ^*T_n).

By remark 2.3.2, ${}_X[T_1 \ldots T_n]$ is a Grothendieck construction and its objects are $(\rho \in {}_X[T_1 \ldots T_{n-1}], \tau \in \rho^*T_n)$. Thus by [1, Theorem 26.8],

$$\operatorname{hocolim}_{{}_X[T_1 \ldots T_n]}U \simeq \operatorname{hocolim}_{\rho \in {}_X[T_1 \ldots T_{n-1}]}\operatorname{hocolim}_{\rho^*T_n}U.$$

On the other hand, by assumption, for all $\rho \in {}_X[T_1 \dots T_{n-1}]$,

$$\mathrm{hocolim}_{\rho^* T_n} U \simeq \text{domain } (\rho).$$

Thus

$$\mathrm{hocolim}_{\rho \in {}_X[T_1 \dots T_{n-1}]} \mathrm{hocolim}_{\rho^* T_n} U \simeq \mathrm{hocolim}_{{}_X[T_1 \dots T_{n-1}]} U.$$

Putting everything together yields $\mathrm{hocolim}_{{}_X[T_1 \dots T_n]} U \simeq \mathrm{hocolim}_{{}_X[T_1 \dots T_{n-1}]} U$ and therefore $\mathscr{F}_*$ is a weak equivalence. □

III. Diagram Commutivity

We will show that the lower left triangle in diagram (8.2) commutes up to homotopy. Recall that the two ways of going around the lower left triangle in diagram (8.2) were obtained by applying the homotopy colimit to

$$\begin{tikzcd} {}_X[RSR] \arrow[r, bend left, "\mathscr{C}^2"] \arrow[r, bend right, "\mathscr{F}^2"'] \arrow[r, phantom, "\Downarrow \eta"] & {}_X[R] \end{tikzcd} \tag{8.3}$$

where η is the natural transformation defined by $\eta_{(\rho,\tau,\gamma)} = \tau \circ \gamma$.

We start by recalling some definitions. Let $\mathcal{M}$ be a category, I be a small category and $D\colon I \to \mathcal{M}$ be a diagram. The *simplicial replacement* of D is the simplicial object $\mathrm{srep}(D)$ of $\mathcal{M}$ defined by

$$\mathrm{srep}(D)_n = \coprod_{(a_0 \leftarrow \cdots \leftarrow a_n) \in I} D(a_n)$$

where the face map $d_i\colon \mathrm{srep}(D)_n \to \mathrm{srep}(D)_{n-1}$ is induced from the following map on $D(a_n)$ indexed by $(a_0 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_n} a_n) \in I$:

- for $i = 0$, $id\colon D(a_n) \to D(a_n)$ where the codomain is indexed by $(a_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_n} a_n)$

- for $0 < i < n$, $id\colon D(a_n) \to D(a_n)$ where the codomain is indexed by $(a_0 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{i-1}} a_{i-1} \xleftarrow{\sigma_i \sigma_{i+1}} a_{i+1} \xleftarrow{\sigma_{i+2}} \cdots \xleftarrow{\sigma_n} a_n)$

- for $i = n$, $D(\sigma_n)\colon D(a_n) \to D(a_{n-1})$ where the codomain is indexed by $(a_0 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_{n-1}} a_{n-1})$

and the degeneracy map $s_i\colon \mathrm{srep}(D)_n \to \mathrm{srep}(D)_{n+1}$ is induced by $id_{D(a_n)}$ where the domain is indexed by $(a_0 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_n} a_n)$ and the codomain is indexed by the chain $(a_0 \xleftarrow{\sigma_1} \cdots \xleftarrow{\sigma_i} a_i \xleftarrow{id} a_i \xleftarrow{\sigma_{i+1}} \cdots \xleftarrow{\sigma_n} a_n)$.

Additionally suppose that J is a small category and $\alpha\colon J \to I$ is a functor. Then we can define $\alpha_{\#}\colon \mathrm{srep}(D\alpha) \to \mathrm{srep}(D)$. Specifically, $\alpha_{\#}$ is induced from $id\colon D\alpha(b_n) \to D(\alpha b_n)$ where the domain is indexed by $(b_0 \xleftarrow{\chi_1} \cdots \xleftarrow{\chi_n} b_n) \in J$ and the codomain is indexed by $(\alpha(b_0) \xleftarrow{\alpha(\chi_1)} \cdots \xleftarrow{\alpha(\chi_n)} \alpha(b_n)) \in I$.

Theorem 8.0.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, $\mathcal{M}$ be a model category and suppose we have a diagram of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\alpha"{name=U}] \arrow[r, bend right, "\beta"'{name=L}] & \mathcal{D} \arrow[r, "F"] & \mathcal{M} \arrow[Rightarrow, from=U, to=L, "\eta"] \end{tikzcd}$$

where η is a natural transformation. Then there exists a map

$$H\colon (\mathrm{srep}(F\alpha)) \times \Delta^1 \to \mathrm{srep}(F)$$

in $s\mathcal{M}$ such that $H_0 = \alpha_\# $ and $H_1 = \beta_\# \circ F\eta$.

Proof. Let I be a category with two objects and one nontrivial morphism between them, specifically, the category $[0 \to 1]$. Since η is a natural transformation, we get an induced functor $\bar{\eta}\colon \mathcal{C} \times I \to \mathcal{D}$ where $\bar{\eta}(X, 0) = \alpha(X)$ and $\bar{\eta}(X, 1) = \beta(X)$.

Let $\{1\}$ be the constant simplicial set whose nth level is 1. Then by inspection, we have the following pushout diagram

$$\begin{array}{ccc}
(\operatorname{srep} F\alpha) \times \{1\} & \xrightarrow{i_1} & (\operatorname{srep} F\alpha) \times \Delta^1 \\
{\scriptstyle F\eta}\downarrow & & \downarrow{\scriptstyle \phi} \\
(\operatorname{srep} F\beta) \times \{1\} & \xrightarrow{j} & \operatorname{srep} F\bar{\eta}
\end{array}$$

where i_1 is the obvious inclusion map induced from the inclusion $\{1\} \to \Delta^1$. Notice that j is an inclusion.

By using $\bar{\eta}_\#\colon \operatorname{srep} F\bar{\eta} \to \operatorname{srep} F$, the composition $\bar{\eta}_\# \circ \phi$ is the desired H. $\square$

A direct application of Theorem 8.0.7 (using $U\colon {}_X[R] \to \mathcal{M}$ for F) to diagram (8.3) yields a "homotopy" at the category level, i.e. yields a morphism

$$H\colon \operatorname{srep}(U \circ \mathscr{C}^2) \times \Delta^1 \to \operatorname{srep}(U) \tag{8.4}$$

such that $H_0 = (\mathscr{C}^2)_\#$ and $H_1 = (\mathscr{F}^2)_\# \circ U\eta$.

Now we move on to getting a useful cylinder object, which involves some categorical lemmas. We start with some notation.

Definition 8.0.8. For an object Y, in some category with coproducts $\mathcal{C}$, and a simplicial set K, we set $Y \odot K$ to be the simplicial object of $\mathcal{C}$ whose nth level is $(Y \odot K)_n = \coprod_{K_n} Y$ with the obvious morphisms.

Lemma 8.0.9. Let $\mathcal{M}$ be a simplicial model category. If Y is an object of $\mathcal{M}$ and K is a simplicial set, then $|Y \odot K| \cong Y \otimes K$

Proof. Let Z be an object of $\mathcal{M}$. We will show that $\mathcal{M}(|Y \odot K|, Z) \cong \mathcal{M}(Y \otimes K, Z)$ and then by Yoneda's Lemma the result will follow. Let Δ be the cosimplicial standard simplex. Then

$$\begin{aligned} \mathcal{M}(|Y \odot K|, Z) &\cong s\mathcal{M}(Y \odot K, Z^{\Delta}) \\ &\cong sSet(K, \mathcal{M}(Y, Z^{\Delta})) \\ &\cong sSet(K, \underline{\mathrm{Map}}(Y, Z)) \\ &\cong \mathcal{M}(Y \otimes K, Z). \end{aligned}$$

□

Lemma 8.0.10. Let $\mathcal{M}$ be a simplicial model category with Reedy cofibrant simplicial object X. Then $|X \times \Delta^1|$ is a cylinder object for $|X|$, meaning that the folding map $id_{|X|} + id_{|X|}$ factors as $|X| \amalg |X| \to |X \times \Delta^1| \xrightarrow{\sim} |X|$.

Proof. To complete this proof, we need to show two things: $|X \times \Delta^1|$ factors the map $|id| + |id| \colon |X| \coprod |X| \to |X|$ and $|X \times \Delta^1| \simeq |X|$.

First, notice that $id + id \colon X \coprod X \to X$ factors through $X \times \Delta^1$ in the obvious way. Then, since realization is a left adjoint and hence preserves colimits, the composite

$$|X| \amalg |X| \cong |X \amalg X| \to |X \times \Delta^1| \to |X|$$

is $|id| + |id|$. Thus showing the first condition.

Second, we will look at $|X \times \Delta^1|$. Let K be the bisimplicial object with level $K_{n,m} = \coprod_{\Delta^1_n} X_m$. Notice that $X \times \Delta^1 = \mathrm{diag}(K)$. Thus

$$|X \times \Delta^1| = |\mathrm{diag}(K)| \cong ||K|_{horiz}|_{vert}$$

where the last isomorphism comes from [13, Lemma on page 94]. Furthermore, $|K|_{horiz} = |X| \odot \Delta^1$ and hence

$$|X \times \Delta^1| = ||X| \odot \Delta^1| \cong |X| \otimes \Delta^1$$

by Lemma 8.0.9. Since $\Delta^1 \to \Delta^0$ is a weak equivalence and $|X|$ is cofibrant by [5, Proposition 3.6], then

$$|X| \otimes \Delta^1 \simeq |X| \otimes \Delta^0 = |X|$$

which completes the proof. $\square$

Now we can return to our "categorical homotopy" (8.4). We will use Lemma 8.0.10 to prove the following theorem, which will shows that our "categorical homotopy" induces a weak equivalence after geometric realization.

Theorem 8.0.11. Let $\mathcal{M}$ be a simplicial model category. If X and Y are simplicial objects in $\mathcal{M}$, X is Reedy cofibrant and there is a morphism $H\colon X \times \Delta^1 \to Y$, then $|H_0|, |H_1|\colon |X| \to |Y|$ are equal in the homotopy category of $\mathcal{M}$.

Proof. We will show that $|H_0|$ and $|H_1|$ are left homotopic, which implies that they are equal in the homotopy category of $\mathcal{M}$. Let $\{i\}$ be the constant simplicial object whose nth level is i. For $i = 0, 1$, H_i is the composition

$$X \cong X \times \{i\} \hookrightarrow X \times \Delta^1 \xrightarrow{H} Y.$$

Thus $|H_i|$ factors through $|H|$ for $i = 0, 1$. Hence $|H_0| + |H_1| \colon |X| \coprod |X| \to |Y|$ factors through $|H|$. Since $|X \times \Delta^1|$ is a cylinder object for $|X|$ (by Lemma 8.0.10), then the factorization of $|H_0| + |H_1|$ through $|H|$ means that $|H_0| + |H_1|$ extends to a map $|X \times \Delta^1| \to |Y|$, i.e. $|H_0|$ and $|H_1|$ are left homotopic. □

Since $\mathrm{srep}(U\mathscr{C}^2)$ is Reedy cofibrant, then by applying Theorem 8.0.11 to morphism (8.4) we obtain:

$$|H_0| = |H_1| \quad \text{in the homotopy category of } \mathcal{M}$$

where

$$|H_0| = |(\mathscr{C}^2)_\#| = \mathscr{C}^2_*, \quad \text{and}$$

$$|H_1| = |(\mathscr{F}^2)_\# \circ U\eta| = \mathscr{F}^2_*.$$

Therefore in diagram (8.2), $\mathscr{C}^2_*$ and $\mathscr{F}^2_*$ commute up to homotopy.

Proof of Theorem 8.0.2

Proof. Let $\mathcal{U}$ be the collection of universal hocolim sieves for the simplicial model category $\mathcal{M}$ with $\mathcal{U}(X)$ the collection of universal hocolim sieves on X. The first two conditions of a Grothendieck topology are easy to check:

For all $f \colon Y \to X$, $f^*(\mathcal{M} \downarrow X) = (\mathcal{M} \downarrow Y)$ and thus in order to prove the first condition, it suffices to show $\mathrm{hocolim}_{(\mathcal{M}\downarrow X)} U \simeq X$. But $(\mathcal{M} \downarrow X)$ has a final object, namely $X \overset{\mathrm{id}}{\to} X$. Thus by [3, Section 6, Lemma 6.8],

$$\mathrm{hocolim}_{(\mathcal{M}\downarrow X)} U \simeq U(id) = X.$$

The second condition automatically follows from the definition of universal hocolim sieve.

The rest of the proof will focus on the third condition. Fix a sieve $S \in \mathcal{U}(X)$ and a sieve R on X such that for all $f \in S$, $f^*R \in \mathcal{U}(\text{domain } f)$. We will show that $R \in \mathcal{U}(X)$.

We start by removing the need to show universality. Up to notation, for any morphism α in $\mathcal{M}$ with codomain X, we have the same assumptions for α^*R as we have for R (when we use α^*S instead of S). In particular, this means that showing R is a hocolim sieve on X will also show (up to notation) that each α^*R is a hocolim sieve. Therefore it suffices to show that R is a hocolim sieve.

We now summarize the discussion from earlier in the section by summarizing the pertinent results about diagram (8.2): in the homotopy category, we have commutative triangles that combine to make a commutative diagram of the form

$$\begin{tikzcd}
\operatorname{hocolim}_{X[R]} U \arrow[r, "\mathscr{F}_*"] & \operatorname{hocolim}_{X[\ ]} U \arrow[r, "\cong"] & X \\
A \arrow[u, "\cong"] \arrow[r] & B. \arrow[u, "\cong"'] \arrow[ul]
\end{tikzcd}$$

By applying $\mathrm{Ho}_{\mathcal{M}}(Z, -)$ (i.e. the homotopy classes of maps in $\mathcal{M}$ from Z to $-$) levelwise to the above diagram, it follows immediately that the diagonal morphism $d_Z\colon \mathrm{Ho}_{\mathcal{M}}(Z, B) \to \mathrm{Ho}_{\mathcal{M}}(Z, \operatorname{hocolim}_{X[R]} U)$ is a bijection. Indeed, the two ways to get from B to X imply that d_Z is an injection whereas the two ways to get from A to $\operatorname{hocolim}_{X[R]} U$ imply that d_Z is a surjection. Since d_Z is a bijection for all Z, then the diagonal map $B \to \operatorname{hocolim}_{X[R]} U$ is an isomorphism. Thus the diagram's commutativity implies that the top horizontal morphism $\mathscr{F}_*$ is also an isomorphism. Hence we have completed the proof of transitivity. $\square$

CHAPTER IX

UNIVERSAL HOCOLIM SIEVES IN THE CATEGORY OF TOPOLOGICAL SPACES

In this section we explore some examples of universal hocolim sieves. Let Δ be the cosimplicial indexing category; in other words, the objects are the sets $[n] = \{0, \dots, n\}$ for $n > 0$ and the morphisms are monotone increasing functions.

Open Covers

Let X be a topological space with open cover $\mathcal{U}$. Set

$$S(\mathcal{U}) \coloneqq \langle \{V \subset X \mid V \in \mathcal{U}\} \rangle.$$

We will show that $S(\mathcal{U})$ is a universal hocolim sieve.

We start by recalling the *Čech complex* $\check{C}(\mathcal{U})_*$ associated to the open cover $\mathcal{U}$. This simplicial set is defined by $\check{C}(\mathcal{U})_n = \coprod V_{a_0} \cap \cdots \cap V_{a_n}$ with the obvious face and degeneracy maps and $V_{a_i} \in \mathcal{U}$ for $i = 0, \dots, n$.

Similarly, the *Čech complex* of a set B will be denoted by $\check{C}(B)_*$. This simplicial set is defined by $\check{C}(B)_n = B^{n+1}$ with the obvious face and degeneracy maps. We remark that $\check{C}(B)_*$ is contracible (see [3, Proposition 3.12 and Example 3.14] and use $f\colon B \to \{*\}$).

Additionally, for a simplicial set K_* we define $\Delta(K_*)$ to be the Grothendieck construction for the functor $\gamma\colon \Delta \to \mathbf{Sets}$ given by $[n] \mapsto K_n$. In particular, $\Delta(K_*)$ is a category with objects $([n], k)$ where $k \in K_n$. We will abuse notation and write k for the object $([n], k)$.

Proposition 9.0.1. For any topological space X and open cover $\mathcal{U}$, $S(\mathcal{U})$ is a universal hocolim sieve.

Proof. Let A be an indexing set for the cover $\mathcal{U}$, i.e. elements of $\mathcal{U}$ take the form V_a for some $a \in A$. Define $\Gamma\colon \Delta(\check{C}(A)_*) \to S(\mathcal{U})$ by $(a_0, \ldots, a_n) \mapsto (V_{a_0} \cap \cdots \cap V_{a_n} \xrightarrow{\iota} X)$ where ι is the inclusion map.

First we show that Γ is a homotopy final functor (as defined by [3]). Indeed, for a fixed $(f\colon Y \to X) \in S(\mathcal{U})$, $(f \downarrow \Gamma)$ is $\Delta(\check{C}(T)_*)$ where $T = \coprod_{V \in \mathcal{U}} (\mathbf{Top} \downarrow X)(Y, V)$ (using Notation 1.0.2) – to see this, notice that any object in $(f \downarrow \Gamma)$ can be viewed (for some n) as an element of

$$\begin{aligned}\coprod_{(a_0,\ldots,a_n)} (\mathbf{Top} \downarrow X)(Y, V_{a_0} \cap \cdots \cap V_{a_n}) &\cong \coprod_{(a_0,\ldots,a_n)} \prod_{i=0}^{n} (\mathbf{Top} \downarrow X)(Y, V_{a_i}) \\ &\cong \prod_{i=0}^{n} \coprod_{V \in \mathcal{U}} (\mathbf{Top} \downarrow X)(Y, V) \\ &= T^{n+1}.\end{aligned}$$

Since $(f\colon Y \to X) \in S(\mathcal{U})$, then f factors through some $V \in \mathcal{U}$ and so T is nonempty. Therefore, the nerve of $\Delta(\check{C}(T)_*)$ is weakly equivalent to $\check{C}(T)_*$, which is itself contracible.

Since Γ is homotopy final, then by [3, "Cofinality Theorem"],

$$\operatorname{hocolim}_{\Delta(\check{C}(A)_*)} U\Gamma \xrightarrow{\simeq} \operatorname{hocolim}_{S(\mathcal{U})} U \to X. \tag{9.1}$$

To see that the composition is a weak equivalence, we use the fact that $\Delta(\check{C}(A)_*)$ is a Grothendieck construction and therefore by [1, Theorem 26.8],

$$\begin{aligned}\mathrm{hocolim}_{\Delta(\check{C}(A)_*)} U\Gamma &\simeq \mathrm{hocolim}_{[n]\in\Delta}\mathrm{hocolim}_{\check{C}(A)_n} U\Gamma \\ &\simeq \mathrm{hocolim}_{\Delta}\check{C}(\mathfrak{U})_*\end{aligned}$$

where the last weak equivalence comes from the fact that $\check{C}(A)_n$ is a discrete category and hence

$$\mathrm{hocolim}_{\check{C}(A)_n} U\Gamma \xrightarrow{\simeq} \varinjlim_{\check{C}(A)_n} U\Gamma = \coprod_{A^{n+1}} V_{a_0} \cap \cdots \cap V_{a_n} = \check{C}(\mathfrak{U})_n.$$

But by [4, Theorem 1.1], hocolim $\check{C}(\mathfrak{U})_* \simeq X$. Therefore, both the left map and the composition in (9.1) are weak equivalences, which implies that the right map is too.

Universality follows immediately from Lemma 2.1.1 and the fact that the pullback on an open cover is an open cover. □

Lastly we remark that $S(\mathfrak{U})$ is an example of a sieve in both the canonical topology (see Example 5.2.4) and the homotopical canonical topology.

Simplices Mapping into X

For topological space X, set

$$\Delta(X) := \{\Delta^n \to X \mid n \in \mathbb{Z}_{\geq 0}\},$$

i.e. all of the maps in $(\mathbf{Top} \downarrow X)$ whose domain is a simplex. We will show that $\langle\Delta(X)\rangle$ is a universal hocolim sieve. First we recall a useful result from [3, Proposition 22.5]:

Proposition 9.0.2. For every topological space X, $\operatorname{hocolim}_{\Delta(X)} U \to X$ is a weak equivalence.

Proposition 9.0.3. Any sieve R on X that contains $\Delta(X)$ is a hocolim sieve.

Proof. Consider the inclusion functor $\alpha \colon \Delta(X) \to R$ and, for each $f \in R$, the natural morphism

$$\chi_f \colon \operatorname{hocolim}_{(\alpha \downarrow f)} U\mu_f \to U(f)$$

where $\mu_f \colon (\alpha \downarrow f) \to R$ is the functor $(i, i \to f) \mapsto i$.

Notice that $(\alpha \downarrow f)$ and $\Delta(\text{domain } f)$ are equivalent categories. Additionally, for all $(i, i \to f) \in (\alpha \downarrow f)$, $U\mu_f(i, i \to f) = \text{domain } i$. Thus

$$\operatorname{hocolim}_{(\alpha \downarrow f)} U\mu_f = \operatorname{hocolim}_{\Delta(\text{domain } f)} U.$$

By Proposition 9.0.2, $\operatorname{hocolim}_{\Delta(\text{domain } f)} U \to (\text{domain } f)$ is a weak equivalence. Hence χ_f is a weak equivalence for all $f \in R$.

The above two paragraphs put us squarely in the hypotheses of [3, Theorem 6.9], which means we may now conclude that

$$\alpha_{\#} \colon \operatorname{hocolim}_{\Delta(X)} U\alpha \to \operatorname{hocolim}_R U$$

is a weak equivalence. Moreover, up to abuse of notation, $U\alpha = U$, which by Proposition 9.0.2 implies that $\operatorname{hocolim}_{\Delta(X)} U\alpha \to X$ is a weak equivalence. Thus in the composition

$$\operatorname{hocolim}_{\Delta(X)} U\alpha \xrightarrow{\alpha_{\#}} \operatorname{hocolim}_R U \to X$$

both the first arrow and the composition itself are weak equivalences. Therefore $\operatorname{hocolim}_R U \to X$ is also a weak equivalence. $\square$

Corollary 9.0.4. For any topological space X, $\langle\Delta(X)\rangle$ is a universal hocolim sieve.

Proof. Let $f\colon Y \to X$ and consider $f^*\langle\Delta(X)\rangle$. Clearly, $\Delta(Y) \subset f^*\langle\Delta(X)\rangle$. Therefore, by Proposition 9.0.3, $f^*\langle\Delta(X)\rangle$ is a hocolim sieve. $\square$

Additionally, we remark that $\langle\Delta(X)\rangle$ is a colim sieve if and only if X is a Delta-generated space. Since not every space is Delta-generated, then for such an X, $\langle\Delta(X)\rangle$ is an example of a sieve in the homotopical canonical topology that is not in the canonical topology.

Corollary 9.0.5. Let $\mathcal{U}$ be an open cover X. Let $R = \langle\{\Delta^n \to V \subset X \mid V \in \mathcal{U}\}\rangle$, i.e. R is generated by the "$\mathcal{U}$-small" simplices. Then R is a universal hocolim sieve.

Proof. We will use the transitivity axiom from the definition of Grothendieck topology with $S(\mathcal{U})$, which by Proposition 9.0.1 is in the homotopical canonical topology. So we only need to show that f^*R is a universal hocolim sieve for every $f \in S(\mathcal{U})$.

Fix $(f\colon Y \to X) \in S(\mathcal{U})$. Then f factors as $Y \xrightarrow{g} W \xrightarrow{i_W} X$ for some $W \in \mathcal{U}$ and inclusion map i_W. Consider $i_W^*R = \langle\{\Delta^n \times_X W \to W \cap V \subset W \mid V \in \mathcal{U}\}\rangle$ (see Lemma 2.1.1). Notice that for any $(\Delta^n \to X) \in R$ that factors through $V \in \mathcal{U}$, $\Delta^n \times_X W \cong \Delta^n \times_V (W \cap V)$ – now we apply the case $V = W$ to see that $\{\Delta^n \to W\}$ is part of i_W^*R's generating set. Therefore $\langle\Delta(W)\rangle \subset i_W^*R$. But by Corollary 9.0.4, $\langle\Delta(W)\rangle$ is in the homotopical canonical topology. Since the homotopical canonical topology is a Grothendieck topology, then any sieve containing a cover is itself a cover. Thus i_W^*R is a universal hocolim sieve. Hence $f^*R = g^*(i_W^*R)$ is a universal hocolim sieve. $\square$

Monogenic Sieves

Recall that a sieve is called monogenic if it can be generated by one morphism. For $f\colon Y \to X$, let $\check{C}(f)_*$ be the *Čech complex* on f. In other words, $\check{C}(f)$ the simplicial object of $\mathcal{M}$ defined by $\check{C}(f)_n = Y \times_X \cdots \times_X Y$, i.e. the pullback of the n-tuple $(Y, \dots, Y)$ over X, with the obvious face and degeneracy maps.

Proposition 9.0.6. For a simplicial model category $\mathcal{M}$, let $S = \langle \{f\colon Y \to X\} \rangle$ be a sieve on X. Then

$$\mathrm{hocolim}_S U \simeq \mathrm{hocolim}\ \check{C}(f)_*.$$

Sketch of Proof. This proof is similar to the proof of Proposition 9.0.1. Basically, $\Gamma\colon \Delta \to S$ defined by $[n] \mapsto (\check{C}(f)_n \to X)$ is homotopy final, which completes the proof. Indeed, for any $(g\colon Z \to X) \in S$, $(g \downarrow \Gamma)$ is $\Delta(\check{C}(K)_*)$ where K is the set $(\mathbf{Top} \downarrow X)\,(Z, Y)$, which is both nonempty and contractible. □

Proposition 9.0.7. If f is locally split, then the sieve generated by f is a universal hocolim sieve.

Proof. Suppose f is a locally split map, i.e. $f\colon Y \to X$ and there is an open cover $\mathcal{U}$ of X such that for all $V \in \mathcal{U}$, $f\big|_{f^{-1}(V)}\colon f^{-1}(V) \to V$ is split. Let $s_V\colon V \to f^{-1}(V)$ be the splitting map for $f\big|_{f^{-1}(V)}$. Then the composition $V \xrightarrow{s_V} f^{-1}(V) \subset Y \xrightarrow{f} X$ equals the inclusion map $V \subset X$ and is in $\langle \{f\} \rangle$. Indeed, $f \circ s_V = id_V$ and the composition clearly factors through f. Thus $(V \subset X) \in \langle \{f\} \rangle$ for all $V \in \mathcal{U}$, which implies that $S(\mathcal{U}) \subset \langle \{f\} \rangle$. Since $S(\mathcal{U})$ is in the homotopical canonical topology (by Proposition 9.0.1), then the Grothendieck topology transitivity axiom implies that any sieve containing it is also in the homotopical canonical topology. Therefore, $\langle \{f\} \rangle$ is in the homotopical canonical topology. □

REFERENCES CITED

[1] Wojciech Chachólski and Jérôme Scherer. *Homotopy theory of diagrams*. Number 736. American Mathematical Soc., 2002.

[2] Brian J Day and G Max Kelly. On topological quotient maps preserved by pullbacks or products. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 67, pages 553–558. Cambridge Univ Press, 1970.

[3] Daniel Dugger. A primer on homotopy colimits. *preprint*, 2008.

[4] Daniel Dugger and Daniel C Isaksen. Hypercovers in topology. *arXiv preprint math/0111287*, 2001.

[5] Paul G Goerss and John F Jardine. *Simplicial homotopy theory*. Springer Science & Business Media, 2009.

[6] Peter T Johnstone. *Sketches of an elephant: A topos theory compendium*, volume 2. Oxford University Press, 2002.

[7] Masaki Kashiwara and Pierre Schapira. Categories and sheaves, volume 332 of. *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 2006.

[8] GM Kelly. Monomorphisms, epimorphisms, and pull-backs. *Journal of the Australian Mathematical Society*, 9(1-2):124–142, 1969.

[9] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[10] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. Springer Science & Business Media, 2012.

[11] J Peter May. *A concise course in algebraic topology*. University of Chicago press, 1999.

[12] Jeffrey Mermin. Three simplicial resolutions. *Progress in commutative algebra*, 1:127–141, 2012.

[13] Daniel Quillen. Higher algebraic k-theory: I. In *Higher K-theories*, pages 85–147. Springer, 1973.

[14] Daniel G Quillen. *Homotopical algebra*, volume 43. Springer, 2006.

[15] Neil P Strickland. The category of cgwh spaces. *preprint*, 12, 2009.